Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 894 (2015) 496–526

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Toroidal orbifolds of $\mathbb{Z}_3$ and $\mathbb{Z}_6$ symmetries of noncommutative tori ☆

Sam Walters

Department of Mathematics & Statistics, University of Northern B.C., Prince George, B.C. V2N 4Z9, Canada

Received 5 February 2015; received in revised form 4 March 2015; accepted 7 March 2015

Available online 13 March 2015

Editor: Stephan Stieberger

Dedicated to George Elliott on his seventieth birthday

Abstract

The Hexic transform ρ of the noncommutative 2-torus A_θ is the canonical order 6 automorphism defined by $\rho(U) = V$, $\rho(V) = e^{-\pi i\theta}U^{-1}V$, where U, V are the canonical unitary generators obeying the unitary Heisenberg commutation relation $VU = e^{2\pi i\theta}UV$. The Cubic transform is $\kappa = \rho^2$. These are canonical analogues of the noncommutative Fourier transform, and their associated fixed point C*-algebras A_θ^ρ, A_θ^κ are noncommutative $\mathbb{Z}_6$, $\mathbb{Z}_3$ toroidal orbifolds, respectively. For a large class of irrationals θ and rational approximations p/q of θ, a projection e of trace $q^2\theta - pq$ is constructed in A_θ that is invariant under the Hexic transform. Further, this projection is shown to be a matrix projection in the sense that it is approximately central, the cut down algebra $eA_\theta e$ contains a Hexic invariant $q \times q$ matrix algebra $\mathcal{M}$ whose unit is e and such that the cut downs eUe, eVe are approximately inside $\mathcal{M}$. It is also shown that these invariant matrix projections are covariant in that they arise from a continuous section $\mathcal{E}(t)$ of C$^\infty$-projections of the continuous field $\{A_t\}_{0<t<1}$ of noncommutative tori C*-algebras such that $\rho(\mathcal{E}(t)) = \mathcal{E}(t)$. It turns out that the projection $\mathcal{E}(t)$ is the support projection of a canonical C$^\infty$-positive element that has the appearance of a noncommutative 2-dimensional Theta function. The topological invariants (or 'quantum' numbers) of $\mathcal{E}(t)$, e, and related projections are computed by a new and quicker method than in previous works. (They would also give topological invariants for finitely generated projective modules over noncommutative orbifolds associated to $\mathbb{Z}_6$ and $\mathbb{Z}_3$ symmetries of noncommutative tori.) We remark that these results have some bearing on research work related to noncommutative orbifolds used in string theory.

☆ Research partly supported by a grant from NSERC.
E-mail address: walters@unbc.ca.
URL: http://hilbert.unbc.ca.

http://dx.doi.org/10.1016/j.nuclphysb.2015.03.008

1. Introduction

For each irrational number θ, the irrational rotation C*-algebra or noncommutative 2-torus A_θ is the unique C*-algebra generated by unitaries U, V enjoying the (unitary) Heisenberg commutation relation

$$VU = \lambda UV, \tag{1.1}$$

where $\lambda = e^{2\pi i\theta}$. By convention we always write $e(x) := e^{2\pi i x}$. As is well-known, if q, p are the position and momentum operators of quantum mechanics satisfying the Heisenberg commutation relation $qp - pq = i\hbar$, the unitaries $U = e^{iq}$, $V = e^{ip}$ satisfy (1.1) with $2\pi\theta = \hbar$.

The (noncommutative) Hexic transform is the order 6 automorphism ρ of A_θ defined by

$$\rho(U) = V, \qquad \rho(V) = \lambda^{-1/2}U^{-1}V \tag{1.2}$$

and the Cubic transform $\kappa = \rho^2$ is its square, the order 3 canonical automorphism κ of A_θ given by[1]

$$\kappa(U) = \lambda^{-1/2}U^{-1}V, \qquad \kappa(V) = U^{-1}. \tag{1.3}$$

The Flip automorphism ϕ is defined by $\phi(U) = U^{-1}$, $\phi(V) = V^{-1}$, and one quickly notes that

$$\rho^2 = \kappa, \qquad \kappa\rho = \phi = \rho^3.$$

These periodic automorphisms arise naturally from the canonical Brenken–Watatani action of $\mathrm{SL}(2,\mathbb{Z})$ on the rotation C*-algebra. They are order 3 and 6 analogues of the (noncommutative) Fourier transform ($V \to U \to V^{-1}$) studied in [7,20–22,24–26]. (The orders 2, 3, 4, 6 are the only orders possible for such canonical automorphisms since these are the only finite orders of matrices in $\mathrm{SL}(2,\mathbb{Z})$. In fact, any finite order automorphism of A_θ induces one on $K_1(A_\theta) = \mathbb{Z}^2$ and hence a matrix in $\mathrm{GL}(2,\mathbb{Z})$.)

The (noncommutative) toroidal orbifolds associated to the symmetry groups $\mathbb{Z}_3$ and $\mathbb{Z}_6$ are the fixed point C*-subalgebras

$$A_\theta^\kappa := \{x \in A_\theta \,|\, \kappa(x) = x\}$$

and A_θ^ρ of A_θ. For the case when θ is *rational* these orbifold algebras take a rather concrete topological form of a 2-sphere $\mathbb{S}^2$ with 3 or 4 singularities[2] each of which takes the form of multiple non-Hausdorff points 'bundled' together.[3] Such realizations make essential use of the

[1] There are other variations of these canonical order 3 automorphisms – e.g., $U \to V \to V^{-1}U^{-1}$. This latter automorphism can, however, be checked to be conjugate to κ by the automorphism $U \to \lambda^{1/6}V^{-1}$, $V \to \lambda^{1/6}U$. So our results extend to these other forms as well with a few appropriate changes.

[2] See [1] for the flip case, and [9–11] for the order 3, 4, 6 cases.

[3] A simpler example is the spectrum of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ (see for example [5, II.2.β]) represented as the closed interval [0, 1] with its ends split in two:

:————————:

realization of the rational noncommutative torus $A_{p/q}$ given in [1] as matrix-valued continuous functions on the unit square that satisfy certain boundary conditions at opposite edges of the square. The finite canonical symmetries would in effect 'fold and paste' the unit square into spheres with singularities (so such orbifolds are sometimes called *noncommutative spheres*).

These non-Hausdorff points can in some way be gleaned from certain projections in a matrix algebra M_q and by the existence of "unbounded" (noncanonical) traces. In the end, these orbifolds in the rational case look like the algebra of all continuous functions $f : \mathbb{S}^2 \to M_q$ which, at the singularities s, commute with certain projections, so that $f(s)$ has a specific block diagonal form at such points.[4] Each of the blocks would correspond to each of the multiple non-Hausdorff points at a given singularity, and by taking the trace of any of the blocks of $f(s)$ one obtains the noncanonical traces. We have detailed out computations of this for the noncommutative Fourier transform in [21] (see Sections 3 and 4).

Thus, in the rational case we have a reasonably good way of visualizing these orbifolds and their singularities – namely, in terms of projections and noncanonical traces. These orbifolds are, however, more difficult to visualize when θ is irrational, but we can nevertheless treat their projections (and modules) in the orbifold as well as the noncanonical trace maps (which still do exist) as indicative of such singularities in a way similar to the rational case.[5]

The projections constructed in this paper and the noncanonical traces (see, for example, Eq. (1.7) below) serve, in part, such purposes for the Cubic and Hexic symmetries.

In [7] it was proved that the orbifold algebras A_θ^κ, A_θ^ρ and their respective (strongly Morita equivalent) C*-crossed products $A_\theta \rtimes_\kappa \mathbb{Z}_3$ and $A_\theta \rtimes_\rho \mathbb{Z}_6$ are approximately finite-dimensional for each irrational θ (extending the Bratteli–Kishimoto Theorem [2] for the Flip case – see also [19]).

In this paper we construct projections that are Hexic (hence Cubic) invariant, show that they have a covariance property (see Theorem 1.2), compute their topological invariants, and we prove that from these projections one obtains many matrix projections (the meaning of which is given in Theorem 1.2 below).

Our first result is the following theorem; it is based on certain topological maps ψ_k, φ_1, φ_{jk} given in Eqs. (1.7) and (1.9)–(1.11) below. We implicitly assume the results of Elliott [8] concerning the continuous field $\{A_t\}_{0<t<1}$ of noncommutative 2-tori.

Theorem 1.1. *There is a continuous section $\mathcal{E} : (0, 1) \to \{A_t\}$ of C^∞-projections of the continuous field $\{A_t\}_{0<t<1}$ of noncommutative 2-tori C*-algebras such that*

(1) $\rho(\mathcal{E}(t)) = \mathcal{E}(t)$, $\kappa(\mathcal{E}(t)) = \mathcal{E}(t)$;
(2) $\mathcal{E}(t)$ *has canonical trace t and κ-topological numbers*

$$\psi_k(\mathcal{E}(t)) = \omega := \frac{1}{2} + \frac{i}{2\sqrt{3}}$$

for $k = 0, 1, 2$;

[4] Perhaps it's worth noting that the number of these singularities is largely independent of the rational parameter θ, though the sizes of the blocks depend on q where $\theta = p/q$.

[5] To appreciate the algebraic difference between the rational and irrational cases we point out that the orbifold in the rational case is a type I C*-algebra (being a C*-subalgebra of matrix-valued continuous functions on a compact Hausdorff space) while the orbifold in the irrational case is a non-type I approximately finite-dimensional algebra as was first shown by Bratteli and Kishimoto [2] for the Flip symmetry.

(3) *$\mathcal{E}(t)$ has ρ-topological numbers*

$$\varphi_1 = 3\omega - 1, \quad \varphi_{20} = \omega, \quad \varphi_{21} = 3\omega, \quad \varphi_{30} = \frac{1}{2}, \quad \varphi_{31} = 2;$$

(4) *$\mathcal{E}(t)$ is the support projection of the noncommutative 2D "Theta function"*

$$\mathbb{X}(t) = t \sum_{m,n} e^{-\frac{\pi t}{\sqrt{3}}(m^2+n^2)} e^{-\pi t(\frac{1}{\sqrt{3}}-i)mn} U_t^n V_t^m \tag{1.4}$$

for $0 < t < 1$; further, $\mathbb{X}(t)$ is positive and $\lim_{t\to 0^+} \|\mathcal{E}(t) - \mathbb{X}(t)\| = 0$.

In obtaining the next result on matrix (or point) projections, we have restricted ourselves to a concrete class $\mathbb{G}$ of irrationals $\theta \in (0, 1)$ (which contains dense G_δ sets in $(0, 1)$) although the result very likely extends to all irrationals (and corresponding convergents) without the restrictions that we impose here.

For example, in the case of the Fourier transform automorphism σ we have shown in [24] that the orbifold A_θ^σ is approximately finite dimensional for a dense G_δ class of irrationals θ, which was later shown to hold for all irrationals in [7]. In addition, we showed that the point-matrix result (in the Fourier case) holds for all irrationals. We made this restriction here in an effort to make the results of Theorem 1.2 accessible (or else the computations will be considerably longer). (Theorem 1.1 is independent of the class $\mathbb{G}$.)

The class $\mathbb{G}$ consists of irrationals θ in the open interval $(0, 1)$ that can be approximated by infinitely many rational numbers $\frac{p}{q}$ (in reduced form) where $p = p_1^2$ is an even perfect square such that[6]

$$0 < \theta - \frac{p}{q} < \frac{0.995}{q^2}. \tag{1.5}$$

It is easy to see that this class $\mathbb{G}$ contains many dense G_δ subsets of $(0, 1)$ and that one can give specific examples of irrational numbers in it. To obtain the projection section referred to in Theorem 1.1, one simply takes $p = 0$, $q = 1$ as we shall see in Sections 3 and 4.

Our second main result is the following matrix projection approximation.

Theorem 1.2. *Let θ be any irrational number in $\mathbb{G}$. Let $\frac{p}{q}$ be a rational number in reduced form (with $p \geq 0$, $q \geq 1$) such that $0 < \theta - \frac{p}{q} < \frac{0.995}{q^2}$, where p is an even perfect square. Then the projection*

$$e = \zeta_{q,\theta}(\mathcal{E}(q^2\theta - pq)) \tag{1.6}$$

in A_θ (has trace $q^2\theta - pq$) is ρ invariant, is approximately central, and there exists a ρ-invariant $q \times q$ matrix algebra $\mathfrak{M} \subset eA_\theta e$ with unit e such that: for any finite subset $F \subset A_\theta$ and each $\epsilon > 0$, there exists large enough q such that exe has distance less than ϵ from $\mathfrak{M}$ for each $x \in F$. (The same conclusions hold for the Cubic transform κ.)

[6] For the approximations in Theorem 1.2, the quantity $q^2\theta - pq$ needs to stay away from 1 – just as in Lemma 7.2 of [24]. The inequality in (1.5) already meets this condition.

The canonical morphism $\zeta_{q,\theta}$ is defined by the relations (3.7).

Recall that the projection e (as a function of the integer parameter q) is approximately central in A_θ if for any finite subset $F \subset A_\theta$ and $\epsilon > 0$ there exists large enough q such that $\|xe - ex\| < \epsilon, \forall x \in F$.

We remark that in the matrix approximation of this theorem, the cut downs of the canonical unitaries eUe, eVe are close to order q unitary matrices of $\mathfrak{M}$. (This is shown in the proof of Section 4 below.) As the generic unitaries $\{U^m V^n\}$ form a total set in the algebra A_θ, the cut down approximation in Theorem 1.2 follows immediately. (In fact, with a little extra work, one can show that they are close to unitary generators of the matrix algebra – as we showed in the Fourier case in [24].)

At the end of Section 5 we compute the topological invariants of the matrix projection e of Theorem 1.2 from those in Theorem 1.1.

The computation of the topological invariants of projections (or, equivalently, finitely generated projective modules, thanks to the Serre–Swan Theorem!) is based on certain noncanonical (and usually unbounded linear functionals) 'twisted' traces defined on the canonical smooth dense *-subalgebra A_θ^∞ (the analogue of the algebra of C^∞ functions on a manifold).

The role that noncommutative tori played in string theory, particularly in compactification of M(atrix) theory, was initiated by Connes, Douglas, and Schwarz [6] and nicely exemplified by Seiberg and Witten in [17] (see, for example their Introduction and Section 6 on gauge theory on a torus). This has lead naturally to the study of orbifolds of noncommutative tori associated to canonical group symmetries.

For instance, in [14] Konechny and Schwarz used K-theoretical topological invariants (obtained by the author in [18]) to study the structure of projective modules over non-commutative $\mathbb{Z}_2$ orbifolds that admit constant curvature Yang–Mills field as well as obtaining their moduli spaces. And in [13], they study moduli spaces of (equivariant) connections with constant curvature on modules over non-commutative even-dimensional tori and on toroidal orbifolds arising from symmetries of the groups $\mathbb{Z}_2$ and $\mathbb{Z}_4$ (with respective actions from the flip and Fourier transform). They give a nice and short summary of this in Section 9 of [12] for the $\mathbb{Z}_2$ orbifold compactification case.

Using the results of the current paper—the construction of the (exotic) projection (or module) $\mathcal{E}(t)$ and its topological invariants[7]—one could solve the associated problems for canonical orbifolds corresponding to the symmetry groups $\mathbb{Z}_3$ and $\mathbb{Z}_6$ of the noncommutative tori (induced by the automorphisms κ, ρ). Namely, the problem of studying (modulo gauge transformations) connections of constant curvature on modules over a non-commutative orbifold, since their classes in the K_0-group are associated to D-brane charges, following Witten's theory [28].

We now explain how we compute these noncanonical trace invariants and recall what they are.

Topological numbers: the continuous field method. In this paper we present a new and quicker method for computing the topological invariants (which are certain quantized complex numbers) of the projections constructed here. What makes this possible is that these projections arise from a continuous field of projections $\mathcal{E} : (0, 1) \to \{A_t\}$, as in Theorem 1.1. The result

[7] The reason we call $\mathcal{E}(t)$ 'exotic' is because it is a fundamental generator of K-theory of the orbifold, much as had been done in [7], and in [20] and [21] in the Fourier transform case.

$\|\mathcal{E}(t) - \mathbb{X}(t)\| \to 0$ in this theorem allows one to compute the topological invariant $\psi(\mathcal{E}) = \psi^t(\mathcal{E}(t))$, which is constant[8] as a function of t (see [7]), by means of the limit

$$\psi(\mathcal{E}) = \lim_{t\to 0^+} \psi^t(\mathbb{X}(t))$$

which will turn out to exist (as computed in Section 5). Here, ψ^t is any of the topological unbounded traces associated with the $\mathbb{Z}_3$ or $\mathbb{Z}_6$ symmetries of the noncommutative tori arising from the Cubic or Hexic transforms. (We will write down these traces shortly.) Once this is done for the section $\mathcal{E}(t)$, we use the covariance relationship that the morphisms $\zeta_{q,\theta}$ have with the unbounded traces ψ (see for instance Eqs. (5.5)–(5.7)) to quickly obtain the topological invariants of projections $e = \zeta_{q,\theta}(\mathcal{E}(q^2\theta - pq))$ in Theorem 1.2. This constitutes what we might call the continuous field method for computing topological invariants, thanks to the noncommutative Theta function $\mathbb{X}(t)$ in (1.4) since their unbounded traces are amenable to direct calculation (which turn out to involve the Theta functions of classical analysis).

The unbounded topological trace maps. Given a (finite order) automorphism β of an algebra A, a (twisted) β-trace is a complex linear map $\psi : A \to \mathbb{C}$ such that

$$\psi(xy) = \psi(\beta(y)x)$$

for all $x, y \in A$. (It is not unlike a KMS state except that we do not have a continuous one-parameter group action.) The restriction of such map ψ to the fixed point subalgebra (orbifold) $A^\beta = \{x \in A \mid \beta(x) = x\}$ defines a trace, so gives an invariant morphism at the K-theory level $\psi_* : K_0(A^\beta) \to \mathbb{C}$.

In the case of noncommutative tori A_θ, such maps are not defined on the whole algebra but on the canonical dense *-subalgebra A_θ^∞ of differentiable elements – namely, Schwartz series $\sum a_{mn}U^m V^n$ where $\{a_{mn}\}$ is rapidly decreasing.

In joint work with Julian Buck [3], we computed such twisted traces for the Cubic transform κ and shown [3, Theorem 3.3] that they form a 3-dimensional complex vector space with basis given by the following basic κ-traces

$$\psi_j^\theta(U^m V^n) = e\big(\tfrac{\theta}{6}(m-n)^2\big)\, \delta_3^{m-n-j} \tag{1.7}$$

where $j = 0, 1, 2$, $VU = e(\theta)UV$, and δ_d^m is the *divisor delta function* given by $\delta_d^m = 1$ if d divides m, and $\delta_d^m = 0$ otherwise. These noncanonical traces induce group homomorphisms on the K_0-group of the Cubic orbifold A_θ^κ which, together with the usual canonical trace state $\boldsymbol{\tau}$, give rise to its Connes–Chern character invariant:

$$T_3 : K_0(A_\theta^\kappa) \to \mathbb{C}^4, \qquad T_3(x) = (\boldsymbol{\tau}(x); \psi_0(x), \psi_1(x), \psi_2(x)).$$

For the identity element, for example, we have $T_3(1) = (1; 1, 0, 0)$. Recall that the canonical trace is defined by $\boldsymbol{\tau}(\sum a_{mn}U^m V^n) = a_{00}$.

For κ-invariant projections (or finitely generated projective modules over the orbifold), their ψ_0, ψ_1, ψ_2 invariants are quantized numbers (shown in the lattice in Fig. 1 below) and they may be called their κ-topological invariants or numbers.

In [4] (using work of Polishchuk [15]) we showed that the homomorphism T_3 is an injective map on $K_0(A_\theta^\kappa) = \mathbb{Z}^8$ in the case that θ is irrational – and in the rational case we would have to

[8] Not unlike the fact that if $f(t)$ is a continuous section of projections of the field $\{A_t\}$ then the label of its canonical trace is constant: $\boldsymbol{\tau}(f(t)) = a + bt$ where a, b are constant integers.

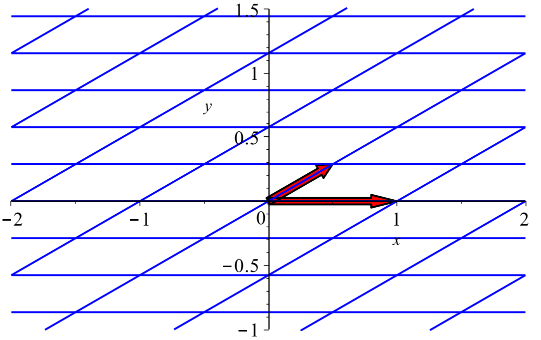

Fig. 1. Lattice points in the complex plane, generated by the vectors 1 and ω, representing permitted κ-topological values of invariant orbifold projections and modules. (It also represents the ρ-topological values with the exception of those of φ_{3j}.)

include Connes' cyclic 2-cocycle that picks out the "label of the trace". Based on the values in Table 2 of [3, (page 37)],[9] the unbounded traces ψ_k of Cubic invariant projections take values in the following lattice in the complex plane

$$\psi_j K_0(A_\theta^\kappa) = \mathbb{Z} + \mathbb{Z}\omega \tag{1.8}$$

(for $j = 0, 1, 2$) where ω is given in Theorem 1.1.[10]

Observe that the range in (1.8) is independent of the "Planck" parameter θ, whereas the canonical bounded trace $\boldsymbol{\tau}$ has range $\boldsymbol{\tau} K_0(A_\theta^\kappa) = \mathbb{Z} + \mathbb{Z}\theta$ which depends on θ.

Remark 1.3. We caution that the "ω" used in Tables 1 and 2 of [3] is not the same as the above ω. If ω' denotes the one used in these tables, which is $\omega' = e(1/6) = \frac{1}{2}(1 + i\sqrt{3})$, then in order to make the conversion from crossed product to fixed point subalgebra, one makes use of the relations $\omega = \frac{1}{\sqrt{3}}\omega'^{1/2}$, $\omega' = 3\omega - 1$.

For the Hexic transform, by Theorem 3.1 [3], there is a unique ρ-trace φ_1^θ (up to scalar multiples) defined on A_θ^∞, a pair of ρ-invariant ρ^2-traces φ_{2j}^θ, and a pair of ρ-invariant ρ^3-traces φ_{3j}^θ given by

$$\varphi_1^\theta(U^m V^n) = e(\tfrac{\theta}{2}(m^2 + n^2)) \tag{1.9}$$

$$\varphi_{20}^\theta(U^m V^n) = e(\tfrac{\theta}{6}(m-n)^2)\delta_3^{m-n}, \qquad \varphi_{21}^\theta(U^m V^n) = e(\tfrac{\theta}{6}(m-n)^2), \tag{1.10}$$

$$\varphi_{30}^\theta(U^m V^n) = e(-\tfrac{\theta}{2}mn)\delta_2^m\delta_2^n, \qquad \varphi_{31}^\theta(U^m V^n) = e(-\tfrac{\theta}{2}mn). \tag{1.11}$$

[9] We point out that the topological invariants listed in Table 2 of [3] are those of the above mentioned maps ψ_k^θ which differ from the maps we used in [3] by normalization constants – particularly for the maps φ_{11}, φ_{12} used in [3] which involved the constants $e(-\frac{\theta}{6})$, $e(-\frac{2\theta}{3})$, respectively, and which should be removed (as we have in fact done so at the end of Section 9 of [3]).

[10] In [3], the unbounded trace values differ by a factor of 3 since we were working with the crossed product C*-algebra $A_\theta \rtimes_\kappa \mathbb{Z}_3$, but since this algebra is strongly Morita equivalent to the fixed point C*-subalgebra A_θ^κ the unbounded trace values in Table 2 of [3] need to be multiplied by 3. For the Hexic case one similarly multiplies the unbounded trace values in Table 1 of [3] by 6.

(Here, of course, $\rho^2 = \kappa$ is the Cubic and $\rho^3 = \phi$ is the Flip.) When no confusion arises we shall simply write $\varphi^{\theta}_{jk} = \varphi_{jk}$. The Connes–Chern character invariant for the Hexic orbifold A^{ρ}_{θ} consists of these together with the canonical trace:

$$T_6 : K_0(A^{\rho}_{\theta}) \to \mathbb{C}^6, \quad T_6(x) = (\boldsymbol{\tau}(x); \varphi_1(x), \varphi_{20}(x), \varphi_{21}(x), \varphi_{30}(x), \varphi_{31}(x))$$

For the identity one has $T_6(1) = (1; 1, 1, 1, 1, 1)$. In this case, the ranges of these noncanonical traces on the K_0-group of the ρ-orbifold subalgebra take the following values

$$\varphi_1 K_0(A^{\rho}_{\theta}) = \mathbb{Z} + \mathbb{Z}\omega, \quad \varphi_{2j} K_0(A^{\rho}_{\theta}) = \mathbb{Z} + \mathbb{Z}\omega, \quad \varphi_{3j} K_0(A^{\rho}_{\theta}) = \tfrac{1}{2}\mathbb{Z}$$

($j = 0, 1$) and are independent of θ. For the canonical trace, $\boldsymbol{\tau} K_0(A^{\rho}_{\theta}) = \mathbb{Z} + \mathbb{Z}\theta$.

Observe the following relations between the maps φ_{2*} and ψ_j

$$\varphi_{20} = \psi_0, \qquad \varphi_{21} = \psi_0 + \psi_1 + \psi_2 \tag{1.12}$$

which will be useful in giving the φ_{2*} topological invariants once we have determined the ψ_j values.

The following order 3 (toral) automorphism commutes with the Cubic transform:

$$\gamma(U) = e(\tfrac{1}{3})U, \quad \gamma(V) = e(-\tfrac{1}{3})V$$

(as well as $\gamma^{-1} = \gamma^2$). (Of all canonical toral $\mathbb{T}^2$-actions on A_θ, only γ, γ^2 and the identity commute with κ.) The relationship between γ and the κ-traces ψ_k is as follows

$$\psi_0\gamma = \psi_0, \quad \psi_1\gamma = e(\tfrac{1}{3})\psi_1, \quad \psi_2\gamma = e(\tfrac{2}{3})\psi_2, \tag{1.13}$$

or $\psi_j\gamma = e(\frac{j}{3})\psi_j$. These enable one to immediately obtain the κ invariants for the corresponding fields $\gamma\mathcal{E}(t)$ and $\gamma^2\mathcal{E}(t)$ in view of (2) of Theorem 1.1:

$$\psi_j(\gamma\mathcal{E}(t)) = e(\tfrac{j}{3})\psi_j(\mathcal{E}(t)) = e(\tfrac{j}{3})\omega \in \{\omega, \omega - 1, 1 - 2\omega\}$$

(the latter values are for $j = 0, 1, 2$, respectively).

Remark 1.4. Note, however, that unlike $\mathcal{E}$, the projection field $\gamma\mathcal{E}$ is not Flip invariant (as the Flip automorphism and γ do not commute) so is not Hexic invariant.

2. Rieffel's bimodule construction and cubic integral transform

In this section we recall the main aspects of Rieffel's equivalence bimodule construction [16] and apply his Theorem 2.15 to our situation with canonical symmetries κ, ρ.

We begin with a locally compact Abelian group M, with $\widehat{M}$ denoting its Pontryagin dual consisting of all characters $M \to \mathbb{T}$, and we form the self-dual direct product group $G = M \times \widehat{M}$. Denote by $\langle\ ,\ \rangle$ the canonical pairing map $M \times \widehat{M} \to \mathbb{T}$ given by $\langle m, s\rangle = s(m)$ where $m \in M$, $s \in \widehat{M}$. The Heisenberg bicharacter on G is the canonical map $\mathfrak{h} : G \times G \to \mathbb{T}$ defined by

$$\mathfrak{h}((m, s), (n, t)) = \langle m, t\rangle$$

for $m, n \in M$, $s, t \in \widehat{M}$.

The Heisenberg projective unitary representation of G is defined by

$$\pi : G \to \mathcal{L}(L^2(M)), \qquad [\pi_{(m,s)} f](n) = \langle n, s\rangle f(n + m)$$

for $m, n \in M$, $s \in \widehat{M}$ and $f \in L^2(M)$. (Here, $L^2(M)$ is the Hilbert space of square-integrable complex functions on M with respect to Haar measure of M.) These unitary operators π_x satisfy the Heisenberg commutation relation

$$\pi_x \pi_y = \mathfrak{h}(x,y)\pi_{x+y} = \mathfrak{h}(x,y)\overline{\mathfrak{h}(y,x)}\pi_y \pi_x \tag{2.1}$$

and their adjoints satisfy

$$\pi_x^* = \mathfrak{h}(x,x)\pi_{-x} \tag{2.2}$$

for $x, y \in G$.

If D is any lattice subgroup of G (i.e., a discrete cocompact subgroup of G), its covolume $|G/D|$ is the Haar measure of a fundamental domain for D in G. The associated twisted group C*-algebra $C^*(D, \mathfrak{h})$ is defined as the C*-subalgebra of the bounded operators on $L^2(M)$ generated by the unitaries π_x for $x \in D$:

$$C^*(D, \mathfrak{h}) = C^*\{\pi_x \,|\, x \in D\}.$$

It is in fact the universal C*-algebra generated by unitaries satisfying the commutation relation (2.1).

The complementary subgroup $D^\perp$, which turns out to also be a lattice subgroup of G (Lemma 3.1 of [16]), is defined by

$$D^\perp = \{y \in G : \mathfrak{h}(x,y)\overline{\mathfrak{h}(y,x)} = 1,\ \forall x \in D\}.$$

By taking the adjoints of the relations in (2.1), we see that the 'dual' unitaries π_y^* for $y \in D^\perp$ satisfy the same relations (2.1) but with the conjugate cocycle $\bar{\mathfrak{h}}$ in place of $\mathfrak{h}$ with the opposite multiplication:

$$\pi_x^* \bullet \pi_y^* = \bar{\mathfrak{h}}(x,y)\pi_{x+y}^* = \bar{\mathfrak{h}}(x,y)\overline{\bar{\mathfrak{h}}(y,x)}\pi_y^* \bullet \pi_x^*, \qquad (\pi_x^*)^* = \bar{\mathfrak{h}}(x,x)\pi_{-x}^* \tag{2.3}$$

for $x, y \in D^\perp$, where we used $\bullet$ for the opposite multiplication ($a \bullet b := ba$). Therefore, the unitaries π_y^*, for $y \in D^\perp$, generate the twisted group C*-algebra $C^*(D^\perp, \bar{\mathfrak{h}})$ but with the understanding that it is now equipped with opposite multiplication:

$$C^*(D^\perp, \bar{\mathfrak{h}}) = C^*\{\pi_y^* \,|\, y \in D^\perp\}.$$

Rieffel's Theorem 2.15 in [16] states that if D is a lattice subgroup of $G = M \times \widehat{M}$, then the completion of the Schwartz space $\mathcal{S}(M)$ of M (under the norm (2.9) defined below) is an equivalence $C^*(D, \mathfrak{h})$-$C^*(D^\perp, \bar{\mathfrak{h}})$-bimodule in a natural way with appropriate C*-inner products. We now recall the module actions and the C*-inner products that accompany this equivalence.

The module actions of the C*-algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \bar{\mathfrak{h}})$ are given by

$$af = \int_D a(x)\pi_x(f)dx = |G/D| \sum_{x\in D} a(x)\pi_x(f) \tag{2.4}$$

$$fb = \int_{D^\perp} b(y)\pi_y^*(f)dy = \sum_{y \in D^\perp} b(y)\pi_y^*(f) \tag{2.5}$$

where $f \in \mathcal{S}(M)$, $a \in C^*(D, \mathfrak{h})$, $b \in C^*(D^\perp, \bar{\mathfrak{h}})$, and where the measure (dx) of each point of D is $|G/D|$ and each point of $D^\perp$ has measure 1. (Here, of course, a is generically represented as $a = \sum_{x\in D} a(x)\pi_x$, where $a(x)$ are its complex coefficients.)

The C*-valued inner products on $\mathcal{S}(M)$ with values in the algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \bar{\mathfrak{h}})$ are given by

$$\langle f, g\rangle_D = |G/D| \sum_{w\in D} \langle f, g\rangle_D(w)\,\pi_w, \qquad \langle f, g\rangle_{D^\perp} = \sum_{z\in D^\perp} \langle f, g\rangle_{D^\perp}(z)\,\pi_z^*$$

where the complex coefficients in these sums are

$$\langle f, g\rangle_D(w_1, w_2) = \langle f, \pi_{(w_1,w_2)} g\rangle_{L^2(M)} = \int_M f(x)\overline{g(x+w_1)}\,\overline{\langle x, w_2\rangle}\,dx \tag{2.6}$$

$$\langle f, g\rangle_{D^\perp}(z_1, z_2) = \int_M \overline{f(x)}\,g(x+z_1)\langle x, z_2\rangle\,dx \tag{2.7}$$

where $(w_1, w_2) \in D$ and $(z_1, z_2) \in D^\perp$. These C*-inner products satisfy the associativity condition

$$\langle f, g\rangle_D h = f\langle g, h\rangle_{D^\perp} \tag{2.8}$$

for all $f, g, h \in \mathcal{S}(M)$. (See [16, pages 266 and 269].) Further, the module actions and C*-inner products satisfy the properties

$$\langle af, g\rangle_D = a\langle f, g\rangle_D, \qquad \langle f, gb\rangle_{D^\perp} = \langle f, g\rangle_{D^\perp} \bullet b = b\langle f, g\rangle_{D^\perp}$$

for $a \in C^*(D, \mathfrak{h})$ and $b \in C^*(D^\perp, \bar{\mathfrak{h}})$, and also for the adjoints of inner products

$$\langle f, g\rangle_D^* = \langle g, f\rangle_D, \quad \langle f, g\rangle_{D^\perp}^* = \langle g, f\rangle_{D^\perp}.$$

The Schwartz space $\mathcal{S}(M)$ gives rise to an equivalence $C^*(D, \mathfrak{h})$–$C^*(D^\perp, \bar{\mathfrak{h}})$ bimodule when it is completed under the norm

$$\|f\| := \|\langle f, f\rangle_D\|^{1/2} = \|\langle f, f\rangle_{D^\perp}\|^{1/2}. \tag{2.9}$$

(The last equality is a theorem of Rieffel – cited in [16].)

Finally, the twisted groups C*-algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \bar{\mathfrak{h}})$ have canonical normalized traces defined on them by

$$\boldsymbol{\tau}_D\left(\sum_{w\in D} a_w\pi_w\right) = a_0, \qquad \boldsymbol{\tau}_{D^\perp}\left(\sum_{z\in D^\perp} b_z\pi_z^*\right) = b_0,$$

($a_w, b_z \in \mathbb{C}$) respectively, and they satisfy the trace equation

$$\boldsymbol{\tau}_D(\langle f, g\rangle_D) = |G/D|\,\boldsymbol{\tau}_{D^\perp}(\langle g, f\rangle_{D^\perp}) \tag{2.10}$$

for all $f, g \in \mathcal{S}(M)$. (See the equation just before Theorem 3.5 in [16].)

From Rieffel's bimodule theorem it follows that if ξ is a Schwartz function such that $\langle \xi, \xi\rangle_{D^\perp} = 1$, then $e = \langle \xi, \xi\rangle_D$ is a projection in $C^*(D, \mathfrak{h})$ of trace $|G/D|$. In this case, one gets an isomorphism

$$\eta : eC^*(D, \mathfrak{h})e \to C^*(D^\perp, \bar{\mathfrak{h}}), \qquad \eta(eae) = \langle \xi, a\xi\rangle_{D^\perp}, \quad \eta^{-1}(b) = \langle \xi b, \xi\rangle_D, \tag{2.11}$$

where $a \in C^*(D, \mathfrak{h})$ and $b \in C^*(D^\perp, \bar{\mathfrak{h}})$. (Note that $e\xi = \xi$ in view of the associative property (2.8) above.)

Self-dual locally compact Abelian groups. We shall now add the symmetry structure to Rieffel's bimodule construction and prove some canonical results from it (see Propositions 2.3 and 2.4 below) that we shall need.

We begin by taking M to be a self-dual locally compact Abelian group, so that we have an isomorphism $\widehat{M} \cong M$ arising from a pairing $\langle\ ,\ \rangle : M \times M \to \mathbb{T}$. We shall require that this pairing be symmetric: $\langle m, n\rangle = \langle n, m\rangle$ for all $m, n \in M$. This in fact holds for the cases that we are interested in, namely the groups $\mathbb{R}$, $\mathbb{Z}_q$, or direct products therefrom. Indeed, as we have recently shown in [27], this symmetric condition can always be arranged for all compactly generated self-dual (locally compact) Abelian groups.

Under this circumstance, we have the Fourier transform $\widehat{f}$ of a Schwartz function $f \in \mathcal{S}(M)$ given by

$$\widehat{f}(s) = \int_M f(x)\overline{\langle x, s\rangle}\, dx$$

for $s \in M$. The square of the Fourier transform gives the flip: $\widehat{\widehat{f}}(t) = f(-t)$.

The self-duality of M permits us to define a canonical order 3 automorphism of $G = M \times M$ by

$$C : G \to G, \quad C(u, v) = (v, -u - v)$$

(for $u, v \in M$) which we call the Cubic map since it will give rise to the Cubic transform automorphism κ given in (1.3). If D is a lattice subgroup of G that is invariant under C (thus, $C(D) = D$), then so is $D^\perp$. In this case, there are corresponding order 3 automorphisms κ and κ' of the group C*-algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \bar{\mathfrak{h}})$, respectively, given by

$$\kappa(\pi_x) = \chi(x)\pi_{Cx}, \qquad \kappa'(\pi_y) = \chi(y)\pi_{Cy}$$

($x \in D$, $y \in D^\perp$) for a suitable "projective character" $\chi : G \to \mathbb{T}$, which we shall soon endeavor to obtain and justify.

To obtain χ, we resort to our recent work [27, Theorem 1.2] where it was shown that for compactly generated locally compact Abelian groups M there exists a projective character $\nu : M \to \mathbb{T}$ — that is, a continuous map (even smooth when M is a Lie group) satisfying the conditions

$$\nu(m + n) = \nu(m)\nu(n)\langle m, n\rangle, \quad \nu(-n) = \nu(n), \quad \nu(0) = 1, \tag{2.12}$$

for $m, n \in M$. Taking $n = -m$ it is clear that $\nu(m)^2 = \langle m, m\rangle$ so that ν is a 'square root' of the quadratic form $\langle m, m\rangle$.

One now defines the map

$$\chi : G \to \mathbb{T}, \quad \chi(r, s) = \overline{\nu(s)} \cdot \overline{\langle r, s\rangle} \tag{2.13}$$

for $(r, s) \in G = M \times M$. It can be checked that this map has the following projective property with respect to the Heisenberg character and the Cubic map C:

$$\chi(x + y) = \chi(x)\chi(y) \cdot \overline{\mathfrak{h}(x, y)}\mathfrak{h}(Cx, Cy) \tag{2.14}$$

for $x, y \in G$. Thereby one obtains the order 3 automorphism κ of the group C*-algebra $C^*(D, \mathfrak{h})$ given by

$$\kappa(\pi_x) = \chi(x)\pi_{Cx} \tag{2.15}$$

for $x \in D$.

The existence of the automorphism κ is established by checking that the universal property of the twisted group C*-algebra $C^*(D, \mathfrak{h})$ — namely that its unitary generators $\{\pi_x\}_{x\in D}$ satisfy the projective commutation relation $\pi_x\pi_y = \mathfrak{h}(x, y)\pi_{x+y}$ for each $x, y \in D$ — remains satisfied when each unitary generator π_x is replaced by the associated unitary operator $\chi(x)\pi_{Cx}$. This is easy to check in view of (2.14) and (2.1).

To check that κ has order 3, it is enough to check that

$$\chi(x)\chi(Cx)\chi(C^2x) = 1$$

for all $x \in G$, and this, again, is straightforward to check. (Here, one makes use of the equation $\nu(m)^2 = \langle m, m\rangle$ for $m \in M$.)

In a similar vain one checks that the Cubic transform is also defined on the Morita equivalent group C*-algebra $C^*(D^\perp, \bar{\mathfrak{h}})$ by

$$\kappa'(\pi^*_y) = \overline{\chi(y)}\pi^*_{Cy} \tag{2.16}$$

for $y \in D^\perp$, and that it has order 3. (Indeed, in the universal property for the unitaries π^*_y (for $y \in D^\perp$) defining the C*-algebra $C^*(D^\perp, \bar{\mathfrak{h}})$, the replacement $\pi^*_y \to \overline{\chi(y)}\pi^*_{Cy}$ preserves the defining commutation relation (2.3), hence the automorphism κ' exists and has order 3.)

As shown in [27], the map ν allows one to define the Cubic transform of Schwartz functions on the group M according to the following prescription

$$f^c(m) := K\nu(m)\widehat{f}(-m) = K\nu(m)\int_M f(t)\langle t, m\rangle dt \tag{2.17}$$

for $m \in M$, where K is a suitable normalizing constant with $|K| = 1$. Theorem 1.3 of [27] says that this transform has order 3. As was shown in the proof of this theorem, $K = \widehat{\nu}(0)^{-1/3}$, and in the two cases that concern us in this paper, for the reals $\mathbb{R}$ the constant is $K = i^{-1/6}$, and in the cyclic group $\mathbb{Z}_q$ case, $K = K_q$ depends on the mod 4 class of q as summarized in Example 2.2 below. It is convenient at this point to give examples of ν and K for the two cases that will concern us here.

Example 2.1. For $M = \mathbb{R}$, the map $\nu(x) = e(\frac{1}{2}x^2)$ satisfies (2.12) with respect to the canonical pairing $\langle x, y\rangle = e(xy) = e^{2\pi ixy}$. As shown in [23], the Cubic transform on the reals then takes the form

$$f^c(x) = i^{-1/6}e(\tfrac{1}{2}x^2)\int_{-\infty}^{\infty} f(t)e(xt)dt$$

where $K = i^{-1/6} = e(-\frac{1}{24})$. (Further, that $f^{ccc} = f$ for each Schwartz function f.) In this paper we shall use the complex Gaussian function $h(x) = e^{-\pi z_1 x^2}$ where $z_1 = \frac{1}{2}(\sqrt{3} - i) = i^{-1/3} = e(-1/12)$ because it is Cubic invariant: $h^c = h$.

Example 2.2. For the cyclic group $M = \mathbb{Z}_q$, one has (as in [27]) the projective character

$$\nu_q(m) = (-1)^m e(\frac{m^2}{2q})$$

with respect to the canonical pairing $\langle m, n\rangle = e(\frac{mn}{q})$. In this case the Cubic transform for cyclic groups takes the form

$$f^c(m) = K_q \nu_q(m)\widehat{f}(-m) = \frac{K_q}{\sqrt{q}}(-1)^m e(\frac{m^2}{2q})\sum_{n=0}^{q-1} f(n)e(\frac{mn}{q})$$

where the normalizing constant K_q has been worked out (see Section 4 of [27]) with the result:

$$K_q = \begin{cases} e(\frac{q-1}{24}) & \text{if } q \text{ is even,} \\ e(\frac{1-q^2}{48}) & \text{if } q \equiv 1 \bmod 4, \\ i^{-1/3}e(\frac{1-q^2}{48}) & \text{if } q \equiv 3 \bmod 4. \end{cases} \tag{2.18}$$

Since in the next section we will be interested in the group $M = \mathbb{R}\times\mathbb{Z}_q\times\mathbb{Z}_q$, we can take the direct product of the projective characters ν of each factor to obtain the projective character on M according to

$$\nu(x, m, n) = e(\tfrac{1}{2}x^2)\nu_q(m)\nu_q(n).$$

This leads to the associated Cubic transform as given by (2.17).

We now prove covariance properties that the Cubic transform has with respect to the module actions and the C*-inner products.

Proposition 2.3. *One has the following bimodule properties for the Cubic transform:*

$$(af)^c = \kappa(a)f^c, \qquad (fb)^c = f^c\kappa'(b)$$

for $a \in C^*(D, \mathfrak{h})$, $b \in C^*(D^\perp, \bar{\mathfrak{h}})$ and for all Schwartz functions $f \in \mathcal{S}(M)$.

Proof. We will check the first one since the second is quite similar. Further, it is enough to check it for $a = \pi_x$ where $x = (x_1, x_2)$. In view of (2.17), we have for any $w \in M$

$$\begin{aligned}
(\pi_x f)^c(w) &= K\nu(w)\int_M (\pi_{(x_1,x_2)}f)(y)\langle y, w\rangle dy \\
&= K\nu(w)\int_M f(y+x_1)\langle y, w+x_2\rangle dy \\
&= \nu(w)\langle -x_1, w+x_2\rangle\cdot K\int_M f(y)\langle y, w+x_2\rangle dy \\
&= \nu(w)\langle -x_1, w+x_2\rangle\overline{\nu(w+x_2)}f^c(w+x_2) \\
&= \langle -x_1, w+x_2\rangle\overline{\nu(x_2)}\overline{\langle w, x_2\rangle}f^c(w+x_2) \\
&= \chi(x_1,x_2)\langle -x_1-x_2, w\rangle f^c(w+x_2) \\
&= [\kappa(\pi_x)f^c](w)
\end{aligned}$$

which ends the proof. $\square$

Proposition 2.4. *One has*

$$\kappa(\langle f, g\rangle_D) = \langle f^c, g^c\rangle_D, \qquad \kappa'(\langle f, g\rangle_{D^\perp}) = \langle f^c, g^c\rangle_{D^\perp}$$

for all Schwartz functions f, g on M.

Proof. We prove the first equality and then show how the second one quickly follows from it and Proposition 2.3. For fixed $(u, v) \in G$, we have

$$\begin{aligned}
\langle f^c, g^c\rangle_D(u, v) &= \int_M f^c(x)\overline{g^c(x+u)}\,\overline{\langle x, v\rangle}dx \\
&= \int_M \nu(x)\widehat{f}(-x)\overline{\nu(x+u)\widehat{g}(-x-u)}\,\overline{\langle x, v\rangle}dx \\
&= \overline{\nu(u)}\int_M \widehat{f}(-x)\overline{\widehat{g}(-x-u)}\,\overline{\langle x, u+v\rangle}dx
\end{aligned}$$

(since the constant K has modulus 1), using inversion $x \to -x$ one gets

$$\begin{aligned}
&= \overline{\nu(u)}\int_M \widehat{f}(x)\overline{\widehat{g}(x-u)}\ \langle x, u+v\rangle dx \\
&= \overline{\nu(u)}\int_M \widehat{f}(x)\overline{\widehat{h}(x)}dx = \overline{\nu(u)}\int_M f(x)\overline{h(x)}dx
\end{aligned}$$

where we used the unitarity of the Fourier transform and h is the function such that

$$\widehat{h}(x) = \widehat{g}(x-u)\overline{\langle x, u+v\rangle}.$$

We compute h by taking the inverse Fourier transform of $\widehat{h}$:

$$\begin{aligned}
h(x) &= \int_M \widehat{h}(t)\langle t, x\rangle dt = \int_M \widehat{g}(t-u)\langle t, x-u-v\rangle dt \\
&= \langle u, x-u-v\rangle \int_M \widehat{g}(t)\langle t, x-u-v\rangle dt \\
&= \langle u, x-u-v\rangle g(x-u-v)
\end{aligned}$$

which gives us h. Therefore, from above we obtain

$$\begin{aligned}
\langle f^c, g^c\rangle_D(u, v) &= \overline{\nu(u)}\int_M f(x)\overline{g(x-u-v)}\,\overline{\langle u, x-u-v\rangle}dx \\
&= \langle f, g\rangle_D(-u-v, u)\cdot\overline{\nu(u)}\langle u, u+v\rangle \\
&= \langle f, g\rangle_D(C^{-1}(u, v))\cdot\overline{\nu(u)}\langle u, u+v\rangle
\end{aligned}$$

This gives

$$\begin{aligned}
\langle f^c, g^c\rangle_D &= |G/D|\sum_{(u,v)\in D}\langle f^c, g^c\rangle_D(u, v)\cdot\pi_{(u,v)} \\
&= |G/D|\sum_{(u,v)\in D}\langle f, g\rangle_D(C^{-1}(u, v))\cdot\overline{\nu(u)}\langle u, u+v\rangle\cdot\pi_{(u,v)}
\end{aligned}$$

which after making the replacement $(u, v) \to (v, -u - v) = C(u, v)$ becomes

$$\langle f^c, g^c\rangle_D = |G/D| \sum_{(u,v)\in D} \langle f, g\rangle_D(u, v) \cdot \overline{\nu(v)}\langle v, -u\rangle \cdot \pi_{C(u,v)}$$

which is indeed equal to

$$\kappa(\langle f, g\rangle_D) = |G/D| \sum_{(u,v)\in D} \langle f, g\rangle_D(u, v) \cdot \kappa(\pi_{(u,v)})$$

by definition of κ since $\kappa(\pi_{(u,v)}) = \chi(u, v)\pi_{C(u,v)} = \overline{\nu(v)}\ \overline{\langle u, v\rangle}\pi_{C(u,v)}$.

We can now see that $\kappa'(\langle f, g\rangle_{D^\perp}) = \langle f^c, g^c\rangle_{D^\perp}$ follows from what we just proved. In view of Proposition 2.3, for each Schwartz function h we have

$$\begin{aligned} h^c\kappa'(\langle f, g\rangle_{D^\perp}) &= (h\langle f, g\rangle_{D^\perp})^c = (\langle h, f\rangle_D g)^c = \kappa(\langle h, f\rangle_D)g^c \\ &= \langle h^c, f^c\rangle_D g^c = h^c\langle f^c, g^c\rangle_{D^\perp}, \end{aligned}$$

hence the result. □

The Flip symmetry. The Flip automorphisms are defined on these twisted group C*-algebras by $\phi(\pi_x) = \pi_{-x}$, $\phi'(\pi_y) = \pi_{-y}$ for $x \in D$, $y \in D^\perp$. One can in fact easily check that they satisfy analogues of Proposition 2.4, namely that

$$\phi(\langle f, g\rangle_D) = \langle \tilde f, \tilde g\rangle_D, \qquad \phi'(\langle f, g\rangle_{D^\perp}) = \langle \tilde f, \tilde g\rangle_{D^\perp}$$

where $\tilde f(x) = f(-x)$. (We also have the analogue of Proposition 2.3 for the Flip.) This, together with Proposition 2.4, gives the result for the Hexic transform as well:

$$\rho(\langle f, g\rangle_D) = \langle \tilde f^{cc}, \tilde g^{cc}\rangle_D, \qquad \rho'(\langle f, g\rangle_{D^\perp}) = \langle \tilde f^{cc}, \tilde g^{cc}\rangle_{D^\perp}$$

where $\tilde f^{cc}$ is the Hexic transform of f (of order 6). (Note that $(\tilde f)^c = \widetilde{f^c}$.)

Therefore, once we have constructed a projection in the inner product form $e = \langle \xi, \xi\rangle_D$ to be Flip and Cubic invariant, it will be invariant under the Hexic transform.

3. Lattice subgroups, C*-inner products, and covariance

Let θ be an irrational number in the class $\mathbb{G}$ and let p/q be a rational approximation, where $p = p_1^2$ is an even perfect square and $q \geq 1$ are relatively prime, satisfying

$$0 < \theta - \frac{p}{q} < \frac{0.995}{q^2}$$

as in (1.5).

Let $M = \mathbb{R} \times \mathbb{Z}_q \times \mathbb{Z}_q$ and consider the lattice D in the self-dual Abelian group $G = M \times M$ ($\cong M \times \widehat{M}$) with basis

$$D: \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \alpha & p_1 & 0 & ; & 0 & 0 & 0 \\ 0 & 0 & 0 & ; & \alpha & p_1 & 0 \end{bmatrix}, \tag{3.1}$$

where $\alpha = \sqrt{\theta - \frac{p}{q}}$. It is clear that D is invariant under the Cubic map $C(u, v) = (v, -u - v)$ on G since $C\varepsilon_1 = -\varepsilon_2$ and $C\varepsilon_2 = \varepsilon_1 - \varepsilon_2$.

Since a fundamental domain for D is $[0, \alpha) \times \mathbb{Z}_q \times \mathbb{Z}_q \times [0, \alpha) \times \mathbb{Z}_q \times \mathbb{Z}_q$, the covolume of D in G is $|G/D| = \alpha^2 q^2 = q^2\theta - pq < 1$. (Recall that the measure of each point of $\mathbb{Z}_q$ is $1/\sqrt{q}$, as in Example 2.2 above.) The C*-algebra $C^*(D,\mathfrak{h})$ is generated by the two unitaries $\pi_{\varepsilon_1}, \pi_{\varepsilon_2}$ satisfying the Heisenberg commutation relation

$$\pi_{\varepsilon_1}\pi_{\varepsilon_2}\pi_{\varepsilon_1}^*\pi_{\varepsilon_2}^* = \mathfrak{h}(\varepsilon_1,\varepsilon_2)\overline{\mathfrak{h}(\varepsilon_2,\varepsilon_1)} = e(\alpha^2 + \tfrac{p_1^2}{q}) = e(\theta) = \lambda.$$

Using (2.15), one checks that

$$\kappa(\pi_{\varepsilon_1}) = \pi_{\varepsilon_2}^{-1}, \quad \kappa(\pi_{\varepsilon_2}) = \lambda^{-1/2}\pi_{\varepsilon_2}^{-1}\pi_{\varepsilon_1}. \tag{3.2}$$

(In the second of these equalities we used the condition that $p = p_1^2$ is even. One can modify the current argument if p is odd.) Therefore, on setting

$$U = \pi_{\varepsilon_2}, \quad V = \pi_{\varepsilon_1},$$

we obtain the desired relations

$$VU = \lambda UV, \quad \kappa(U) = \lambda^{-1/2}U^{-1}V, \quad \kappa(V) = U^{-1}.$$

The complementary lattice $D^\perp$ can easily be checked to be generated by the following basis vectors

$$D^\perp: \begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{bmatrix} = \begin{bmatrix}\beta & -cp_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta & -cp_1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

where

$$\beta := \frac{1}{q\alpha} > 1$$

and c is an integer such that $cp \equiv 1 \mod q$. We have

$$C(\delta_1) = -\delta_2, \quad C(\delta_2) = \delta_1 - \delta_2, \quad C(\delta_3) = -\delta_4, \quad C(\delta_4) = \delta_3 - \delta_4.$$

We consider the associated unitary generators

$$V_j := \pi_{\delta_j}^* = \pi_{-\delta_j}$$

for $j = 1,2,3,4$ which can be checked to satisfy the commutation relations

$$V_1V_2 = e(\theta')V_2V_1, \qquad V_3V_4 = e(\tfrac{1}{q})\,V_4V_3, \qquad V_jV_k = V_kV_j, \qquad V_3^q = V_4^q = I, \tag{3.3}$$

for $j = 1,2$ and $k = 3,4$, where $\theta' = \beta^2 + \frac{c^2p}{q} = \frac{c^2p\theta + d'}{q\theta - p}$ (for some integer d') is in the usual $\mathrm{GL}(2,\mathbb{Z})$ orbit of θ. The C*-algebra $C^*(D^\perp,\mathfrak{h})$ is generated by the unitaries V_1, V_2, V_3, V_4 and is isomorphic to $M_q(A_{\theta'})$.

In view of (2.16), one checks that the Cubic transform κ' on $C^*(D^\perp,\bar{\mathfrak{h}})$ is given on these unitary generators as follows

$$\kappa'(V_1) = V_2^{-1}, \qquad \kappa'(V_2) = e(-\tfrac{1}{2}\theta')V_2^{-1}V_1, \tag{3.4}$$

$$\kappa'(V_3) = V_4^{-1}, \qquad \kappa'(V_4) = -e(-\tfrac{1}{2q})V_4^{-1}V_3. \tag{3.5}$$

We now consider the Schwartz function f on M defined by

$$f(x,n,m) = \frac{3^{1/8}}{\sqrt{q}}h(x)\varphi(n,m), \quad h(x) = e^{-\pi z_1 x^2}, \quad \varphi(n,m) = e(\tfrac{nm}{q}),$$

where, $z_1 = \frac{1}{2}(\sqrt{3} - i) = i^{-1/3} = e(-1/12)$. We have $h^c = h$, though φ is not Cubic invariant but instead satisfies $\varphi^c = \varphi W_0$ for some unitary matrix W_0, as shown by Lemma 4.2 below. It will then follow that $f^c = f W_0$.

We now compute the inner product $\langle f, f\rangle_D$. Its series expansion coefficients are

$$\begin{aligned}
&\langle f, f\rangle_D(m\varepsilon_1 + n\varepsilon_2) \\
&\quad = \langle f, f\rangle_D(m\alpha, mp_1, 0;\ n\alpha, np_1, 0) \\
&\quad = \frac{1}{q}\sum_{r,s=0}^{q-1}\int_{\mathbb{R}} f(x,r,s)\overline{f(x+m\alpha, r+mp_1, s)}e(-xn\alpha)e(-\tfrac{rnp_1}{q})\,dx \\
&\quad = \frac{1}{q^2}3^{1/4}\sum_{r,s=0}^{q-1}\int_{\mathbb{R}} h(x)\varphi(r,s)\overline{h(x+m\alpha)\varphi(r+mp_1,s)}e(-xn\alpha)e(-\tfrac{rnp_1}{q})\,dx \\
&\quad = \frac{1}{q^2}3^{1/4}\int_{\mathbb{R}} h(x)\overline{h(x+m\alpha)}e(-xn\alpha)dx \cdot \sum_{r,s=0}^{q-1}\varphi(r,s)\overline{\varphi(r+mp_1,s)}e(-\tfrac{rnp_1}{q}) \\
&\quad = \frac{1}{q^2}3^{1/4}\overline{H(m\alpha, n\alpha)}\cdot\Omega
\end{aligned}$$

where

$$H(a,b) = \int_{\mathbb{R}} \overline{h(x)}h(x+a)e(bx)\,dx = 3^{-1/4}e^{-\pi(\frac{1}{\sqrt{3}}+i)ab}e^{-\frac{\pi}{\sqrt{3}}(a^2+b^2)} \tag{3.6}$$

(note: $\frac{1}{\sqrt{3}} + i = \frac{2i}{\sqrt{3}}z_1$) and

$$\Omega = \sum_{r,s=0}^{q-1}\varphi(r,s)\overline{\varphi(r+mp_1,s)}e(-\tfrac{rnp_1}{q}) = \sum_{r,s=0}^{q-1} e(-\tfrac{smp_1}{q})e(-\tfrac{rnp_1}{q}) = q^2\delta_q^m\delta_q^n.$$

Thus,

$$\langle f, f\rangle_D(m\varepsilon_1 + n\varepsilon_2) = 3^{1/4}\overline{H(m\alpha, n\alpha)}\delta_q^m\delta_q^n$$

and (since $|G/D| = q^2\alpha^2$),

$$\begin{aligned}
\langle f, f\rangle_D &= |G/D|\sum_{m,n}\langle f, f\rangle_D(m\varepsilon_1 + n\varepsilon_2)\pi_{m\varepsilon_1+n\varepsilon_2} \\
&= 3^{1/4}q^2\alpha^2\sum_{m,n}\overline{H(m\alpha, n\alpha)}\delta_q^m\delta_q^n\pi_{m\varepsilon_1+n\varepsilon_2} \\
&= 3^{1/4}q^2\alpha^2\sum_{m,n}\overline{H(mq\alpha, nq\alpha)}\pi_{mq\varepsilon_1+nq\varepsilon_2}.
\end{aligned}$$

From $\pi_{mq\varepsilon_1+nq\varepsilon_2} = U^{qn}V^{qm}$ and inserting the expression for H from (3.6), we obtain

$$\langle f, f\rangle_D = q^2\alpha^2\sum_{m,n}e^{-\pi(\frac{1}{\sqrt{3}}-i)q^2\alpha^2 mn}e^{-\frac{\pi}{\sqrt{3}}q^2\alpha^2(m^2+n^2)}U^{qn}V^{qm}.$$

We now explain that the positive element $\langle f, f\rangle_D$ is both invariant and covariant.

To show that the element $\langle f, f\rangle_D$ has covariant form, consider the continuous cross section $\mathbb{X}$ of the continuous field of rotation C*-algebras $\{A_t\}_{0<t<1}$ (obtained by replacing $q^2\alpha^2$ by a free parameter t) defined by

$$\mathbb{X}(t) = t \sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} e^{-\frac{\pi}{\sqrt{3}}t(m^2+n^2)} U_t^n V_t^m$$

(for $0 < t < 1$) which in fact consists of smooth elements in A_t^∞. (Basically, $\mathbb{X}(t)$ is the above expression for $\langle f, f\rangle_D$ after setting $p = 0$, $q = 1$, and $q^2\alpha^2 = t$.) Letting $\theta_q := q^2\theta - pq = q^2\alpha^2$, and defining the homomorphism $\zeta = \zeta_{q,\theta} : A_{\theta_q} \to A_\theta$ by

$$\zeta(U_{\theta_q}) = U_\theta^q, \quad \zeta(V_{\theta_q}) = V_\theta^q, \tag{3.7}$$

one immediately gets the covariance relation

$$\langle f, f\rangle_D = \zeta(\mathbb{X}(\theta_q)). \tag{3.8}$$

(For our purposes, the covariance of an element simply means that it arises from a continuous section of the field of C*-algebras $\{A_t\}$ in this manner.)

Further, with κ_0 denoting the Cubic transform of A_{θ_q}, one has the commutative diagram

$$\begin{tikzcd} A_{\theta_q} \arrow[r, "\zeta"] \arrow[d, "\kappa_0"'] & A_\theta \arrow[d, "\kappa"] \\ A_{\theta_q} \arrow[r, "\zeta"'] & A_\theta \end{tikzcd}$$

so that

$$\kappa\zeta = \zeta\kappa_0. \tag{3.9}$$

It is straightforward to check that $\mathbb{X}(t)$ is Cubic invariant (for each t), therefore it immediately follows that $\langle f, f\rangle_D$ is invariant under the Cubic transform κ.

Another way to see the invariance is using Proposition 2.4 and Lemma 4.2 below (which gives $f^c = fW_0$):

$$\kappa\langle f, f\rangle_D = \langle f^c, f^c\rangle_D = \langle fW_0, fW_0\rangle_D = \langle f, f\rangle_D.$$

Further, by inspection $\mathbb{X}(t)$ and $\langle f, f\rangle_D$ are invariant under the Flip ϕ (since f is an even function), therefore they are invariant under the Hexic transform $\rho = \phi\kappa^2$. It now follows that the support projections of $\mathbb{X}(t)$ and $\langle f, f\rangle_D$ (after they are shown to exist below) are ρ invariant as well.

4. The matrix projection

In this section we carry out C*-inner product computations that show:

(i) $\langle f, f\rangle_{D^\perp}$ is invertible, so that the projection e of Theorem 1.2 exists,
(ii) that e is approximately central,
(iii) that the cut downs eUe, eVe are approximated by unitary matrices in a matrix algebra that is ρ invariant and whose identity is e.

(The approximations hold for large q.)

To this end, we start by computing the inner product $\langle f, U^r f\rangle_{D^\perp} = \langle f, \pi_{\varepsilon_2}^r f\rangle_{D^\perp}$ where $r = 0, 1$ is fixed. With $r = 0$ this will give existence and approximate centrality of the projection, and with $r = 1$ we get the matrix approximations for eUe, eVe.

Let $g = \pi_{\varepsilon_2}^r f = \pi_{r\varepsilon_2} f$ so that

$$\begin{aligned} g(y, j, k) &= (\pi_{r\varepsilon_2} f)(y, j, k) = (\pi_{(0,0,0;r\alpha,rp_1,0)} f)(y, j, k) = e(r\alpha y)e(\tfrac{rp_1 j}{q}) f(y, j, k) \\ &= \frac{3^{1/8}}{\sqrt{q}} e(r\alpha y)e(\tfrac{rp_1 j}{q}) h(y)\varphi(j, k) \end{aligned}$$

and

$$\begin{aligned} &\langle f, \pi_{\varepsilon_2}^r f\rangle_{D^\perp}(\Sigma_j n_j\delta_j) = \langle f, g\rangle_{D^\perp}(n_1\beta, -n_1cp_1, n_3;\ n_2\beta, -n_2cp_1, n_4) \\ &= \int\limits_{\mathbb{R}\times\mathbb{Z}_q\times\mathbb{Z}_q} \overline{f(x, n, m)} g(x + n_1\beta, n - n_1cp_1, m + n_3) e(xn_2\beta + \tfrac{-n_2cp_1n+mn_4}{q})\, dx dn dm \\ &= \frac{1}{q}\sum_{m,n=0}^{q-1} e(\tfrac{-n_2cp_1n+mn_4}{q}) \int\limits_{\mathbb{R}} \overline{f(x, n, m)} g(x + n_1\beta, n - n_1cp_1, m + n_3) e(xn_2\beta)\, dx \\ &= \frac{3^{1/8}}{q\sqrt{q}} \sum_{m,n=0}^{q-1} e(\tfrac{-n_2cp_1n}{q}) e(\tfrac{mn_4}{q}) \\ &\quad \times \int\limits_{\mathbb{R}} \overline{h(x)\varphi(n, m)} g(x + n_1\beta, n - n_1cp_1, m + n_3) e(xn_2\beta)\, dx \end{aligned}$$

The integral here is

$$\begin{aligned} &\frac{3^{1/8}}{\sqrt{q}} \int\limits_{\mathbb{R}} \overline{h(x)\varphi(n, m)} e(r\alpha(x + n_1\beta)) e(\tfrac{rp_1(n-n_1cp_1)}{q}) h(x + n_1\beta) \\ &\quad \times \varphi(n - n_1cp_1, m + n_3)\ e(xn_2\beta)\, dx \\ &= \frac{3^{1/8}}{\sqrt{q}} \int\limits_{\mathbb{R}} \overline{h(x)} e(-\frac{nm}{q}) e(r\alpha(x + n_1\beta)) \\ &\quad \times e(\tfrac{rp_1(n-n_1cp_1)}{q}) h(x + n_1\beta) e(\frac{(n - n_1cp_1)(m + n_3)}{q})\ e(xn_2\beta)\, dx \\ &= \frac{3^{1/8}}{\sqrt{q}} e(\frac{(nn_3 - n_1cp_1m - n_1cp_1n_3)}{q}) e(\tfrac{rp_1(n-n_1cp_1)}{q})\ e(r\alpha n_1\beta) \\ &\quad \times \int\limits_{\mathbb{R}} \overline{h(x)} h(x + n_1\beta)\ e(x(n_2\beta + r\alpha))\, dx \\ &= \frac{3^{1/8}}{\sqrt{q}} e(\frac{(nn_3 - n_1cp_1m - n_1cp_1n_3)}{q}) e(\tfrac{rp_1n-rn_1cp}{q})\ e(\frac{rn_1}{q})\ H(n_1\beta, n_2\beta + r\alpha) \end{aligned}$$

and since $cp \equiv 1 \bmod q$ we get

$$= \frac{3^{1/8}}{\sqrt{q}} e(\frac{(nn_3 - n_1cp_1m - n_1cp_1n_3)}{q}) e(\tfrac{rp_1n}{q}) \;\; H(n_1\beta, n_2\beta + r\alpha)$$
$$= \frac{3^{1/8}}{\sqrt{q}} e(-\tfrac{n_1cp_1n_3}{q}) e(-\tfrac{n_1cp_1m}{q}) e(\tfrac{(n_3+rp_1)n}{q}) \;\; H(n_1\beta, n_2\beta + r\alpha)$$

Hence

$$\langle f, \pi^r_{\varepsilon_2} f\rangle_{D^\perp}(\Sigma_j n_j\delta_j)$$
$$= \frac{3^{1/4}}{q^2} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{cp_1n_3n_1}{q}) \sum_{m,n=0}^{q-1} e(\tfrac{-n_2cp_1n}{q}) e(\tfrac{mn_4}{q}) e(-\tfrac{n_1cp_1m}{q}) e(\tfrac{(n_3+rp_1)n}{q})$$
$$= \frac{3^{1/4}}{q^2} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{cp_1n_3n_1}{q}) \sum_{m,n=0}^{q-1} e(\tfrac{(n_4-n_1cp_1)}{q} m) e(\tfrac{(n_3+rp_1-n_2cp_1)}{q} n)$$
$$= 3^{1/4} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{cp_1n_3n_1}{q}) \; \delta_q^{n_4-n_1cp_1} \delta_q^{n_3+rp_1-n_2cp_1}.$$

This gives the C*-inner product

$$\langle f, \pi^r_{\varepsilon_2} f\rangle_{D^\perp} = \sum_{n_1n_2n_3n_4} \langle f, \pi^r_{\varepsilon_2} f\rangle_{D^\perp}(\Sigma n_j\delta_j) \cdot \pi^*_{\Sigma n_j\delta_j}$$
$$= 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha)$$
$$\times \sum_{n_3,n_4=0}^{q-1} e(-\tfrac{cp_1n_3n_1}{q}) \; \delta_q^{n_4-n_1cp_1} \delta_q^{n_3+rp_1-n_2cp_1} \cdot \pi^*_{\Sigma n_j\delta_j}$$

and here we can just substitute $n_3 = n_2cp_1 - rp_1$ in the complex exponential and pull it out of the summation because of the delta functions:

$$= 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{cp_1n_1(n_2cp_1-rp_1)}{q})$$
$$\times \sum_{n_3,n_4=0}^{q-1} \delta_q^{n_4-n_1cp_1} \delta_q^{n_3+rp_1-n_2cp_1} \cdot \pi^*_{\Sigma n_j\delta_j}.$$

Working out $\pi^*_{\Sigma n_j\delta_j}$, we get

$$\pi_{\Sigma n_j\delta_j} = \pi_{(n_2\delta_2+n_4\delta_4)+(n_1\delta_1+n_3\delta_3)} = \pi_{n_2\delta_2+n_4\delta_4}\pi_{n_1\delta_1+n_3\delta_3} = \pi_{n_2\delta_2}\pi_{n_4\delta_4}\pi_{n_1\delta_1}\pi_{n_3\delta_3}$$

and

$$\pi^*_{\Sigma n_j\delta_j} = V_3^{n_3} V_1^{n_1} V_4^{n_4} V_2^{n_2}$$

so the inner product becomes

$$\langle f, \pi^r_{\varepsilon_2} f\rangle_{D^\perp} = 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{cp_1^2n_1(n_2c-r)}{q})$$
$$\times \sum_{n_3,n_4=0}^{q-1} \delta_q^{n_4-n_1cp_1} \delta_q^{n_3+rp_1-n_2cp_1} \cdot V_3^{n_3} V_4^{n_4} V_1^{n_1} V_2^{n_2}$$

$$= 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(-\tfrac{c^2 p n_1 n_2}{q}) e(\tfrac{cpn_1 r}{q}) e(\theta' n_1 n_2)$$
$$\times \sum_{n_3,n_4=0}^{q-1} \delta_q^{n_4 - n_1 cp_1} \delta_q^{n_3 + rp_1 - n_2 cp_1} \cdot V_3^{n_3} V_2^{n_2} V_4^{n_4} V_1^{n_1}$$

inserting $\theta' = \beta^2 + \frac{c^2 p}{q}$ and using $cp \equiv 1 \bmod q$ gives

$$= 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(\tfrac{rn_1}{q}) e(\beta^2 n_1 n_2)$$
$$\times \sum_{n_3,n_4=0}^{q-1} \delta_q^{n_4 - n_1 cp_1} \delta_q^{n_3 + rp_1 - n_2 cp_1} \cdot V_3^{n_3} V_2^{n_2} V_4^{n_4} V_1^{n_1}$$
$$= 3^{1/4} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(\tfrac{rn_1}{q}) e(\beta^2 n_1 n_2) \cdot V_3^{n_2 cp_1 - rp_1} V_2^{n_2} V_4^{n_1 cp_1} V_1^{n_1}$$
$$= 3^{1/4} V_3^{-rp_1} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(\tfrac{rn_1}{q}) e(\beta^2 n_1 n_2) \cdot (V_3^{cp_1} V_2)^{n_2} (V_4^{cp_1} V_1)^{n_1}$$
$$= 3^{1/4} V_3^{-rp_1} \sum_{n_1,n_2} H(n_1\beta, n_2\beta + r\alpha) e(\tfrac{rn_1}{q}) e(\beta^2 n_1 n_2) \cdot W_2^{n_2} W_1^{n_1}$$

where the unitaries

$$W_1 = V_4^{cp_1} V_1, \quad W_2 = V_3^{cp_1} V_2$$

satisfy the unitary Heisenberg commutation relation

$$W_1 W_2 = e(\beta^2) W_2 W_1.$$

We have obtained

$$\langle f, \pi_{\varepsilon_2}^r f \rangle_{D^\perp} = V_3^{-rp_1} \sum_{m,n} 3^{1/4} H(m\beta, n\beta + r\alpha) e(\tfrac{rm}{q}) \; W_1^m W_2^n$$

where

$$3^{1/4} H(m\beta, n\beta + r\alpha) = e^{-\pi(\frac{1}{\sqrt{3}} + i)\beta^2 m(n + r\alpha\beta^{-1})} e^{-\frac{\pi}{\sqrt{3}}\beta^2 [m^2 + (n + r\alpha\beta^{-1})^2]}.$$

Taking $r = 0, 1$ one gets

$$\langle f, f \rangle_{D^\perp} = \sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}} + i)\beta^2 mn} e^{-\frac{\pi}{\sqrt{3}}\beta^2 (m^2 + n^2)} W_1^m W_2^n \tag{4.1}$$

and

$$\langle f, \pi_{\varepsilon_2} f \rangle_{D^\perp} = \langle f, Uf \rangle_{D^\perp} = V_3^{-p_1} \widetilde{Y} \tag{4.2}$$

where

$$\widetilde{Y} := \sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}} + i)\beta^2 m(n + \alpha\beta^{-1})} e^{-\frac{\pi}{\sqrt{3}}\beta^2 [m^2 + (n + \alpha\beta^{-1})^2]} e(\tfrac{m}{q}) W_1^m W_2^n. \tag{4.3}$$

Using techniques similar to those used in Section 6 of [24] one checks that

$$Y := \langle f, f \rangle_{D^\perp} = \sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}}+i)\beta^2 mn} e^{-\frac{\pi}{\sqrt{3}}\beta^2(m^2+n^2)} W_1^m W_2^n$$

is invertible in $C^*(D^\perp, \bar{\mathfrak{h}})$ for any $\beta^2 > 1$. We do however wish to present a much shorter and quicker proof that $\langle f, f \rangle_{D^\perp}$ is invertible for $\beta^2 > 1.005$ (which suffices for the purposes of Theorem 1.2) as shown by the following lemma.

Lemma 4.1. *For any pair of unitaries W_1, W_2 in a unital C^*-algebra, the element*

$$Y(x) = \sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}}+i)xmn} e^{-\frac{\pi}{\sqrt{3}}x(m^2+n^2)} W_1^m W_2^n$$

is invertible for $x > 1.005$.

Proof. It suffices to check that $\|Y(x) - I\| < 1$ for $x > 1.005$. We have

$$Y(x) - I = \sum_{(m,n)\neq(0,0)} e(-\tfrac{x}{2}mn) e^{-\frac{\pi}{\sqrt{3}}x(m^2+mn+n^2)} W_1^m W_2^n$$

and

$$\|Y(x) - I\| \le \sum_{(m,n)\neq(0,0)} e^{-\frac{\pi}{\sqrt{3}}x(m^2+mn+n^2)} = \sum_{m,n} e^{-\frac{\pi}{\sqrt{3}}x(m^2+mn+n^2)} - 1 = \Theta(x) - 1$$

where

$$\begin{aligned}\Theta(x) &= \sum_m e^{-\frac{\pi}{\sqrt{3}}xm^2} \sum_n e^{-\frac{\pi}{\sqrt{3}}x(n^2+mn)} = \sum_m e^{-\frac{\pi}{\sqrt{3}}xm^2} e^{\frac{\pi}{4\sqrt{3}}xm^2} \sum_n e^{-\frac{\pi}{\sqrt{3}}x(n+\frac{m}{2})^2} \\ &= \sum_m e^{-\frac{\pi\sqrt{3}}{4}xm^2} \sum_n e^{-\frac{\pi}{\sqrt{3}}x(n+\frac{m}{2})^2}\end{aligned}$$

(here break the sum over m according to parity, $m \to 2m$ and $m \to 2m+1$)

$$\begin{aligned}&= \sum_m e^{-\pi\sqrt{3}xm^2} \sum_n e^{-\frac{\pi}{\sqrt{3}}xn^2} + \sum_m e^{-\pi\sqrt{3}x(m+\frac{1}{2})^2} \sum_n e^{-\frac{\pi}{\sqrt{3}}x(n+\frac{1}{2})^2} \\ &= \vartheta_3(0, i\sqrt{3}x)\vartheta_3(0, \tfrac{i}{\sqrt{3}}x) + \vartheta_2(0, i\sqrt{3}x)\vartheta_2(0, \tfrac{i}{\sqrt{3}}x).\end{aligned}$$

(Here we used the classical Theta functions which we recall at the beginning of Section 5.) Note that $\Theta(x)$ is a decreasing function as each series is clearly a decreasing function. One checks by direct computation that $\Theta(1.005) = 1.9987\ldots$ so that for $x > 1.005$ one has $\|Y(x) - I\| \le \Theta(x) - 1 < C_0 < 1$ where $C_0 := \Theta(1.005) - 1 = 0.9987\ldots$, hence the result. □

We therefore conclude that the inner product $Y = \langle f, f \rangle_{D^\perp}$ is positive, invertible, and

$$\|Y - I\| < C_0 < 1 \tag{4.4}$$

whenever $\beta^2 > 1.005$, where $C_0 = \Theta(1.005) - 1 = 0.9987\ldots$ is the constant stated in the preceding proof. As θ is in the G_δ class $\mathbb{G}$, so that $0 < q(q\theta - p) = \frac{1}{\beta^2} < \frac{1}{1.005} = 0.99502\ldots$, Eq. (4.4) is satisfied and therefore the above inner product $\langle f, f \rangle_{D^\perp}$ is invertible.

The projection. Normalizing f by the positive invertible element

$$b := \langle f, f\rangle_{D^\perp}^{-1/2}$$

by forming the Schwartz function $\xi := fb$, so that $\langle \xi, \xi\rangle_{D^\perp} = 1$, we obtain the desired projection

$$e = \langle \xi, \xi\rangle_D \tag{4.5}$$

of Theorem 1.2. Further, this projection is the support projection of $\langle f, f\rangle_D$ as can be seen quickly by checking that $\langle f, f\rangle_D \langle fb^2, fb^2\rangle_D = e$.

As a result, we have the isomorphism

$$\eta : eA_\theta e \to C^*(D^\perp, \bar{\mathfrak{h}}) \tag{4.6}$$

of unital C*-algebras defined by

$$\eta(x) = \langle \xi, x\xi\rangle_{D^\perp}, \quad \eta^{-1}(y) = \langle \xi y, \xi\rangle_D$$

for $x \in eA_\theta e$ and $y \in C^*(D^\perp, \bar{\mathfrak{h}})$. (It is worthwhile remembering here that when operators acting on the right come out of the inner product $\langle\ ,\ \rangle_{D^\perp}$, they do so with the *opposite* multiplication (opposite of the usual multiplication of operators on the Hilbert space $L^2(M)$). Thus, $\langle g_1, g_2 a\rangle_{D^\perp} = \langle g_1, g_2\rangle_{D^\perp} \bullet a = a\langle g_1, g_2\rangle_{D^\perp}$.)

In view of the fact that $\langle f, f\rangle_D$ is ρ-invariant (discussed near the end of the previous section), it follows that e is ρ-invariant. In view of the trace equation (2.10), the projection e has trace

$$\boldsymbol{\tau}(e) = \boldsymbol{\tau}\langle \xi, \xi\rangle_D = |G/D|\boldsymbol{\tau}'\langle \xi, \xi\rangle_{D^\perp} = q^2\theta - pq.$$

In addition, the covariance relation (3.8) immediately gives the covariance equation (1.6) for e.

Approximate centrality. We now prove that the projection e is approximately central in A_θ – and for this it suffices to check that e approximately commutes with U since upon applying κ (or ρ) to the approximation $Ue \approx eU$ (noting that e is ρ and κ invariant) one sees that e also approximately commutes with V.

As $-\|H\|I \le H \le \|H\|I$ for any Hermitian operator H, taking $H = Y - I$ and using $\|Y - I\| < C_0$ as we just obtained in (4.4), we get[11]

$$-C_0 I \le Y - I \le C_0 I \quad \text{or} \quad (1 - C_0)I \le Y \le (1 + C_0)I. \tag{4.7}$$

Taking the "η^{-1}" of this and noting that $\eta(e) = I$ and $\eta(\langle f, f\rangle_D) = \langle f, f\rangle_{D^\perp}$ we get

$$(1 - C_0)e \le \langle f, f\rangle_D \le (1 + C_0)e.$$

Writing $x := \langle f, f\rangle_D$, so that x^{-1} denotes its inverse in $eA_\theta e$, one has

$$\tfrac{1}{1+C_0} e \le x^{-1} \le \tfrac{1}{1-C_0} e, \quad \|x^{-1}\| \le \tfrac{1}{1-C_0} < 834$$

whenever $\beta^2 > 1.005$ (or, equivalently, whenever $0 < q(q\theta - p) < 0.99502\ldots$ as noted above).

In virtually the same way we have done in [24] (see Section 7) one checks $\|xU - Ux\| \to 0$ as $q \to \infty$, hence writing

$$eU - Ue = x^{-1}(xU - Ux) + (xU - Ux)x^{-1} + x^{-1}(Ux^2 - x^2U)x^{-1}$$

we get

[11] Note that $\|Y\| > 1$ and $\|Y^{-1}\| > 1$. For if $\|Y\| \le 1$, then $Y \le I$ and $I - Y$ would be a positive element in an irrational rotation C*-algebra of zero trace, so $Y = I$, which is not the case. The same contradiction arises if $\|Y^{-1}\| \le 1$.

$$\|eU - Ue\| \le 2\|x^{-1}\|\|xU - Ux\| + \|x^{-1}\|^2\|Ux^2 - x^2U\|$$
$$\le 1668\|xU - Ux\| + 834^2\|Ux^2 - x^2U\|$$

which goes to 0 for large q. One concludes that the projection e is therefore approximately central: the commutator norms $\|eU - Ue\|$ and $\|eV - Ve\|$ are arbitrarily small for large q. This proves that e is approximately central as per Theorem 1.2.

We now work out the cut down approximation in Theorem 1.2. For this, we need a lemma.

Lemma 4.2. *There exists a unitary matrix W_0 in $C^*(V_3, V_4) \cong M_q$ such that $f^c = fW_0$ and $\xi^c = \xi W_0$. Further, W_0 satisfies $\kappa'^2(W_0)\kappa'(W_0)W_0 = I$.*

Proof. Much as we have done in Section 4 of [24], one removes the first and fourth columns of the group G and from the lattice subgroup D in (3.1), and correspondingly removes the first and fourth columns as well as the δ_1, δ_2 row vectors in $D^\perp$. These result in lattice subgroups D_0, $D_0^\perp$ of the finite group $G_0 = M_0 \times M_0$ where $M_0 = \mathbb{Z}_q \times \mathbb{Z}_q$. With this in mind, for the discrete factor φ of f, one computes that $\langle\varphi, \varphi\rangle_{D_0} = qI$ and $\langle\varphi, \varphi\rangle_{D_0^\perp} = qI$ are scalars. Therefore, by normalizing $\varphi_0 := \frac{1}{\sqrt{q}}\varphi$ and taking

$$W_0 = \langle\varphi_0, \varphi_0^c\rangle_{D_0^\perp} \tag{4.8}$$

which is a unitary matrix in $C^*(V_3, V_4) \cong M_q$, one has

$$\varphi_0^c = \langle\varphi_0, \varphi_0\rangle_{D_0}\varphi_0^c = \varphi_0\langle\varphi_0, \varphi_0^c\rangle_{D_0^\perp} = \varphi_0 W_0$$

whence $f^c = fW_0$ since $f = h \otimes \varphi$ and $h^c = h$. Using Proposition 3.9 we get

$$\kappa'(b^{-2}) = \kappa'\langle f, f\rangle_{D^\perp} = \langle f^c, f^c\rangle_{D^\perp} = \langle fW_0, fW_0\rangle_{D^\perp} = W_0^* \bullet \langle f, f\rangle_{D^\perp} \bullet W_0$$
$$= W_0 b^{-2} W_0^*$$

whence $\kappa'(b) = W_0 b W_0^*$. Therefore, by Proposition 2.3 this gives

$$\xi^c = (fb)^c = f^c\kappa'(b) = (fW_0)\kappa'(b) = f(\kappa'(b)W_0)$$
$$= f(W_0 b) = (fb)W_0 = \xi W_0$$

as claimed. The equation $\kappa'^2(W_0)\kappa'(W_0)W_0 = I$ is easy to verify by taking the Cubic transform on the equation $\varphi_0^c = \varphi_0 W_0$ twice, thereby obtaining

$$\varphi_0 = (((\varphi_0^c)^c)^c = \varphi_0[W_0 \bullet \kappa'(W_0) \bullet \kappa'^2(W_0)],$$

which, upon taking the inner product via $\langle\varphi_0, \cdot\rangle_{D_0^\perp}$, gives the result. $\square$

Next we check that

$$\eta\kappa = \kappa''\eta \tag{4.9}$$

where $\kappa'' = W_0^*\kappa'(\ )W_0$. Note that the matrix algebra $\mathcal{M}_q = C^*(V_3, V_4)$ is κ''-invariant since it is κ'-invariant (in view of (3.5)) and W_0 is a unitary in $\mathcal{M}_q$.

For each $x \in eA_\theta e$ we have, using $\xi = \xi^c W_0^*$ by Lemma 4.2,

$$\eta\kappa(x) = \langle\xi, \kappa(x)\xi\rangle_{D^\perp} = \langle\xi^c W_0^*, \kappa(x)\xi^c W_0^*\rangle_{D^\perp} = W_0^*\langle\xi^c, \kappa(x)\xi^c\rangle_{D^\perp} W_0$$
$$= W_0^*\langle\xi^c, (x\xi)^c\rangle_{D^\perp} W_0 = W_0^*\kappa'(\langle\xi, x\xi\rangle_{D^\perp})W_0 = \kappa''\eta(x).$$

Cut down approximation. From Eq. (4.2) (and noting that $e\xi = \xi = fb$), the first cut down approximation is

$$\begin{aligned}\eta(eUe) &= \langle \xi, eUe\xi\rangle_{D^\perp} = \langle \xi, U\xi\rangle_{D^\perp} = b\langle f, Uf\rangle_{D^\perp} b = bV_3^{-p_1}\widetilde{Y}b \\ &= Y^{-1/2}V_3^{-p_1}\widetilde{Y}Y^{-1/2}.\end{aligned}$$

Here we use the limit approximation

$$\|V_3^{p_1}YV_3^{-p_1} - Y\| \to 0$$

as $q \to \infty$, which can be checked exactly as in [24, (Section 8)]. As the norm of Y is bounded above and below (as we saw above in (4.7)) we get

$$\|V_3^{p_1}Y^{-1/2} - Y^{-1/2}V_3^{p_1}\| \to 0.$$

Further, since $\|\widetilde{Y} - Y\| \to 0$ for large q, so that $\|Y^{-1/2}\widetilde{Y}Y^{-1/2} - I\| \to 0$ (since the norms of $Y, Y^{-1/2}, Y^{-1}$ are all bounded), we obtain the desired cut down approximation

$$\eta(eUe) \approx V_3^{-p_1}Y^{-1/2}\widetilde{Y}Y^{-1/2} \approx V_3^{-p_1}$$

by a unitary in the matrix algebra (in fact, one of its unitary generators).

The cut down approximation for $\eta(eVe)$ by a unitary matrix in $\mathcal{M}_q$ now follows from Eq. (4.9) since

$$\kappa''\eta(eVe) = \eta\kappa(eVe) = \eta(eU^*e) \approx V_3^{p_1}$$

hence

$$\eta(eVe) \approx \kappa''^{-1}(V_3^{p_1}) \in \mathcal{M}_q.$$

Hence, $\eta(eVe)$ is approximately in $\kappa''^{-1}(\mathcal{M}_q) = \mathcal{M}_q$.

It's not hard to check that the isomorphism η commutes with the Flip:

$$\eta\phi = \phi'\eta$$

since f and ξ are even functions. As the Hexic transform is $\rho = \phi\kappa^2$, it follows that the approximating matrix C*-algebra $\mathcal{M}_q$ is invariant under the Hexic transform also – and, as we noted earlier, that the point projection $e = \langle \xi, \xi\rangle_D$ is invariant under ρ as well.

This completes the proof of Theorem 1.2.

5. Computation of topological invariants

To facilitate the computation of the topological invariants mentioned in Theorem 1.1 based on the continuous field method that we outlined in the Introduction, we recall the classical Theta functions

$$\vartheta_2(z,s) = \sum_n e^{\pi is(n+\frac{1}{2})^2} e^{i2z(n+\frac{1}{2})}, \tag{5.1}$$

$$\vartheta_3(z,s) = \sum_n e^{\pi isn^2} e^{i2zn}, \tag{5.2}$$

$$\vartheta_4(z,s) = \sum_n (-1)^n e^{\pi isn^2} e^{i2zn}, \tag{5.3}$$

for $z, s \in \mathbb{C}$ and $\mathrm{Im}(s) > 0$, where all summations range over the integers. (For a wonderful treatment see [29], whose definitions we adopt.) We further recall the following inversion formulas for Theta functions which we will use:

$$\vartheta_3(z,s) = (-is)^{-1/2}\, e^{z^2/(\pi i s)}\, \vartheta_3\left(\tfrac{z}{s}, -\tfrac{1}{s}\right),$$
$$\vartheta_2(z,s) = (-is)^{-1/2}\, e^{z^2/(\pi i s)}\, \vartheta_4\left(\tfrac{z}{s}, -\tfrac{1}{s}\right). \tag{5.4}$$

We begin with the computation of φ_1 on the projection $\mathcal{E}(t)$ by working out $\varphi_1^t(\mathbb{X}(t))$ and then taking its limit as $t \to 0^+$. For this we first need a lemma.

Lemma 5.1. *For any complex numbers A, B with $\mathrm{Re}(B) > 0$, one has*

$$\lim_{t\to 0^+} \sqrt{t}\,\vartheta_k(tA, itB) = \frac{1}{\sqrt{B}}$$

for $k = 2, 3$.

Proof. The proof is a simple consequence of the inversion formula for Theta functions. Applying (5.4) one gets

$$\sqrt{t}\,\vartheta_3(tA, itB) = \sqrt{t}\,\frac{1}{\sqrt{tB}} e^{-tA^2/(\pi B)} \vartheta_3(\tfrac{A}{iB}, \tfrac{i}{tB}) = \frac{1}{\sqrt{B}} e^{-tA^2/(\pi B)} \vartheta_3(\tfrac{A}{iB}, \tfrac{i}{tB}).$$

As $t \to 0^+$, the ϑ_3 factor here goes to 1, hence the result. The same proof holds for the limit involving ϑ_2, the only difference is that the inversion formula converts ϑ_2 into ϑ_4, and the limit of $\vartheta_4(-\frac{iA}{B}, \frac{i}{tB})$ is still 1. □

Let us now proceed to compute $\varphi_1^t(\mathbb{X}(t))$. We have

$$\begin{aligned}
\varphi_1^t(\mathbb{X}(t)) &= t\sum_{m,n} e^{-\frac{\pi}{\sqrt{3}}t(m^2+n^2)} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} \varphi_1^t(U_t^n V_t^m) \\
&= t\sum_{m,n} e^{-\frac{\pi}{\sqrt{3}}t(m^2+n^2)} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} e^{i\pi t(m^2+n^2)} \\
&= t\sum_{m,n} e^{i\pi(1+\frac{i}{\sqrt{3}})t(m^2+n^2)} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} \\
&= t\sum_{n} e^{i\pi(1+\frac{i}{\sqrt{3}})tn^2} S_n
\end{aligned}$$

where the sum over m here is

$$\begin{aligned}
S_n &= \sum_m e^{i\pi(1+\frac{i}{\sqrt{3}})tm^2} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} \\
&= \vartheta_3(i\tfrac{\pi}{2}(\tfrac{1}{\sqrt{3}} - i)tn,\ (1 + \tfrac{i}{\sqrt{3}})t)
\end{aligned}$$

using the inversion formula here gives

$$= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{\frac{1}{\sqrt{3}} - i}} e^{\frac{\pi}{4}(\frac{1}{\sqrt{3}}-i)tn^2} \vartheta_3(\tfrac{\pi}{2}n, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})}).$$

This gives

$$\varphi_1^t(\mathbb{X}(t)) = \frac{\sqrt{t}}{\sqrt{\frac{1}{\sqrt{3}} - i}} \sum_n e^{i\pi(1+\frac{i}{\sqrt{3}})tn^2} e^{\frac{\pi}{4}(\frac{1}{\sqrt{3}}-i)tn^2} \vartheta_3(\tfrac{\pi}{2}n, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})})$$

$$= \frac{\sqrt{t}}{\sqrt{\frac{1}{\sqrt{3}} - i}} \sum_n e^{\frac{i\pi}{4}(3+i\sqrt{3})tn^2} \vartheta_3(\tfrac{\pi}{2}n, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})})$$

here, the ϑ_3 factor depends only on the parity of n (as ϑ_3 has period π in the first variable), so the sum breaks down according to parity as

$$= \frac{\sqrt{t}}{\sqrt{\frac{1}{\sqrt{3}} - i}} \left\{ \sum_n e^{i\pi(3+i\sqrt{3})tn^2} \vartheta_3(0, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})}) + \sum_n e^{i\pi(3+i\sqrt{3})t(n+\frac{1}{2})^2} \vartheta_3(\tfrac{\pi}{2}, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})}) \right\}$$

$$= \frac{\sqrt{t}}{\sqrt{\frac{1}{\sqrt{3}} - i}} \left\{ \vartheta_3(0, (3+i\sqrt{3})t)\, \vartheta_3(0, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})}) + \vartheta_2(0, (3+i\sqrt{3})t)\, \vartheta_3(\tfrac{\pi}{2}, \tfrac{-1}{t(1+\frac{i}{\sqrt{3}})}) \right\}.$$

Using Lemma 5.1 we obtain

$$\varphi_1(\mathcal{E}) = \lim_{t\to 0^+} \varphi_1^t(\mathbb{X}(t)) = \frac{2}{\sqrt{\frac{1}{\sqrt{3}} - i}} \frac{1}{\sqrt{\sqrt{3} - 3i}} = \frac{1 + i\sqrt{3}}{2} = 3\omega - 1$$

which gives the value stated in Theorem 1.1.

Remark 5.2. We carried out computations with the computer algebra software Maple to verify this and later results for these limits.

We next compute and show that $\psi_k(\mathcal{E}) = \omega$ for $k = 0, 1, 2$. We have

$$\psi_k^t(\mathbb{X}(t)) = t \sum_{m,n} e^{-\pi t(\frac{1}{\sqrt{3}}-i)mn} e^{-\frac{\pi t}{\sqrt{3}}(m^2+n^2)} \psi_k^t(U_t^n V_t^m)$$

$$= t \sum_{m,n} e^{-\pi t(\frac{1}{\sqrt{3}}-i)mn} e^{-\frac{\pi t}{\sqrt{3}}(m^2+n^2)} e(\tfrac{t}{6}(m-n)^2)\, \delta_3^{m-n-k}$$

the replacement $m \to n + k + 3m$ gives

$$= t \sum_{m,n} e^{-\pi t(\frac{1}{\sqrt{3}}-i)(n+k+3m)n} e^{-\frac{\pi t}{\sqrt{3}}((n+k+3m)^2+n^2)} e(\tfrac{t}{6}(k+3m)^2)$$

$$= t \sum_{m} e^{-\frac{\pi t}{\sqrt{3}}(k+3m)^2} e(\tfrac{t}{6}(k+3m)^2) N(m)$$

where (after simplifying exponents)

$$N(m) = \sum_n e^{-\pi t(\frac{1}{\sqrt{3}}-i)[n^2+(k+3m)n]} e^{-\frac{\pi t}{\sqrt{3}}[2n^2+2n(k+3m)]} = \sum_n e^{-\pi t c n^2} e^{-\pi t c(k+3m)n}$$

where $c = \sqrt{3} - i$ (has positive real part)

$$= \vartheta_3(i\tfrac{\pi}{2}tc(k+3m), itc)$$

which, under inversion (5.4), becomes

$$N(m) = \frac{1}{\sqrt{tc}} e^{\frac{\pi}{4}tc(k+3m)^2} \vartheta_3(\tfrac{\pi}{2}k + \tfrac{3\pi}{2}m, \tfrac{i}{tc}) = \frac{1}{\sqrt{tc}} e^{\frac{\pi}{4}tc(k+3m)^2} \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc})$$

since ϑ_3 has period π in the first variable and, as we noted earlier, that the ϑ_3 factor here depends only on the parity of m. Therefore, we obtain

$$\begin{aligned}
\psi_k^t(\mathbb{X}(t)) &= \frac{t}{\sqrt{tc}} \sum_m e^{-\frac{\pi t}{\sqrt{3}}(k+3m)^2} e(\tfrac{t}{6}(k+3m)^2) e^{\frac{\pi}{4}tc(k+3m)^2} \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc}) \\
&= \frac{\sqrt{t}}{\sqrt{c}} \sum_m e^{-\frac{\pi t}{12}(\sqrt{3}+3i)(k+3m)^2} e(\tfrac{t}{6}(k+3m)^2) \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc}) \\
&= \frac{\sqrt{t}}{\sqrt{c}} e^{-\frac{\pi t}{12}(\sqrt{3}+3i)k^2} e(\tfrac{t}{6}k^2) \\
&\quad \cdot \sum_m e^{-\frac{\pi t}{12}(\sqrt{3}+3i)(9m^2+6km)} e(\tfrac{t}{6}(9m^2+6km)) \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc}) \\
&= \frac{\sqrt{t}}{\sqrt{c}} e^{-\frac{\pi t}{12}(\sqrt{3}+3i)k^2} e(\tfrac{t}{6}k^2) \\
&\quad \cdot \sum_m e^{-\frac{\pi t}{4}(\sqrt{3}+3i)(3m^2+2km)} e(\tfrac{t}{2}(3m^2+2km)) \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc}) \\
&= \frac{\sqrt{t}}{\sqrt{c}} e^{-\frac{\pi t}{12}(\sqrt{3}+3i)k^2} e(\tfrac{t}{6}k^2) \sum_m e^{i\pi 3btm^2} e^{2i(\pi bkt)m} \vartheta_3(\tfrac{\pi}{2}k + \tfrac{\pi}{2}m, \tfrac{i}{tc})
\end{aligned}$$

where we wrote $b := \frac{1}{4}(1 + i\sqrt{3}) = \frac{i}{4}c$; summing according to parity of m gives

$$\begin{aligned}
&= \frac{\sqrt{t}}{\sqrt{c}} e^{-\frac{\pi t}{12}(\sqrt{3}+3i)k^2} e(\tfrac{t}{6}k^2) \\
&\quad \cdot \Bigg\{ \sum_m e^{i\pi 12btm^2} e^{2i(2\pi bkt)m} \vartheta_3(\tfrac{\pi}{2}k, \tfrac{i}{tc}) \\
&\quad + \sum_m e^{i\pi 12bt(m+\frac{1}{2})^2} e^{2i(2\pi bkt)(m+\frac{1}{2})} \vartheta_4(\tfrac{\pi}{2}k, \tfrac{i}{tc}) \Bigg\} \\
&= \frac{\sqrt{t}}{\sqrt{c}} e^{-\frac{\pi t}{12}(\sqrt{3}+3i)k^2} e(\tfrac{t}{6}k^2) \\
&\quad \cdot \left\{ \vartheta_3(2\pi bkt, 12bt) \vartheta_3(\tfrac{\pi}{2}k, \tfrac{i}{tc}) + \vartheta_2(2\pi bkt, 12bt) \vartheta_4(\tfrac{\pi}{2}k, \tfrac{i}{tc}) \right\}.
\end{aligned}$$

Taking the limit of this in view of Lemma 5.1 we get

$$\psi_k(\mathcal{E}) = \lim_{t \to 0^+} \psi_k^t(\mathbb{X}(t)) = 2 \frac{1}{\sqrt{c}} \frac{1}{\sqrt{3c}} = \frac{2}{c\sqrt{3}} = \omega.$$

(Here, both $\vartheta_3(\frac{\pi}{2}k, \frac{i}{tc})$, $\vartheta_4(\frac{\pi}{2}k, \frac{i}{tc})$ have limit 1 as $t \to 0^+$.) Thus, we can write down the Connes–Chern κ-topological invariant for the projection field $\mathcal{E}(t)$ as

$$T_3(\mathcal{E}(t)) = (t; \omega, \omega, \omega).$$

Applying the automorphism γ to this projection we get, in view of Eqs. (1.13), its invariants as well

$$T_3(\gamma\mathcal{E}(t)) = (t; \omega, e(\tfrac{1}{3})\omega, e(-\tfrac{1}{3})\omega) = (t; \omega, \omega - 1, 1 - 2\omega)$$
$$T_3(\gamma^2\mathcal{E}(t)) = (t; \omega, e(\tfrac{2}{3})\omega, e(-\tfrac{2}{3})\omega) = (t; \omega, 1 - 2\omega, \omega - 1)$$

From Eq. (1.12) we therefore obtain

$$\varphi_{20}(\mathcal{E}) = \omega, \quad \varphi_{21}(\mathcal{E}) = 3\omega.$$

We now compute $\varphi_{3k}(\mathcal{E})$:

$$\varphi_{3k}^t(\mathbb{X}(t)) = t\sum_{m,n} e^{-\pi(\frac{1}{\sqrt{3}}-i)tmn} e^{-\frac{\pi}{\sqrt{3}}t(m^2+n^2)} \varphi_{3k}^t(U_t^n V_t^m).$$

When $k = 0$ we get

$$\begin{aligned}
\varphi_{30}^t(\mathbb{X}(t)) &= t\sum_{m,n} e^{-\frac{\pi}{\sqrt{3}}tmn} e^{-\frac{\pi}{\sqrt{3}}t(m^2+n^2)} \delta_2^m \delta_2^n \\
&= t\sum_{m,n} e^{-\frac{4\pi}{\sqrt{3}}tmn} e^{-\frac{4\pi}{\sqrt{3}}t(m^2+n^2)} \\
&= t\sum_{m,n} e^{-\frac{4\pi}{\sqrt{3}}t(m^2+mn+n^2)} \\
&= t\sum_{m,n} e^{-\frac{4\pi}{\sqrt{3}}t[(m+\frac{n}{2})^2+\frac{3}{4}n^2]} \\
&= t\sum_{n} e^{-\frac{3\pi t}{\sqrt{3}}n^2} \sum_{m} e^{-\frac{4\pi}{\sqrt{3}}t(m+\frac{n}{2})^2}
\end{aligned}$$

which can be written as two sums depending on the parity of n as follows

$$\begin{aligned}
&= t\left\{\sum_n e^{-\frac{12\pi t}{\sqrt{3}}n^2} \sum_m e^{-\frac{4\pi t}{\sqrt{3}}m^2} + \sum_n e^{-\frac{12\pi t}{\sqrt{3}}(n+\frac{1}{2})^2} \sum_m e^{-\frac{4\pi t}{\sqrt{3}}(m+\frac{1}{2})^2}\right\} \\
&= t\left\{\vartheta_3(0, \tfrac{12it}{\sqrt{3}})\vartheta_3(0, \tfrac{4it}{\sqrt{3}}) + \vartheta_2(0, \tfrac{12it}{\sqrt{3}})\vartheta_2(0, \tfrac{4it}{\sqrt{3}})\right\}.
\end{aligned}$$

Now apply Lemma 5.1 to obtain the limit of this

$$\varphi_{30}(\mathcal{E}) = 2\frac{1}{\sqrt{\frac{12}{\sqrt{3}}}}\frac{1}{\sqrt{\frac{4}{\sqrt{3}}}} = \frac{1}{2}.$$

For $k = 1$ we similarly get

$$\begin{aligned}
\varphi_{31}^t(\mathbb{X}(t)) &= t\sum_{m,n} e^{-\frac{\pi t}{\sqrt{3}}(m^2+mn+n^2)} = t\sum_{m,n} e^{-\frac{\pi t}{\sqrt{3}}[(m+\frac{n}{2})^2+\frac{3}{4}n^2]} \\
&= t\sum_n e^{-\frac{3\pi t}{4\sqrt{3}}n^2} \sum_m e^{-\frac{\pi t}{\sqrt{3}}(m+\frac{n}{2})^2}
\end{aligned}$$

$$= t \left\{ \sum_n e^{-\frac{3\pi t}{\sqrt{3}} n^2} \sum_m e^{-\frac{\pi t}{\sqrt{3}} m^2} + \sum_n e^{-\frac{3\pi t}{\sqrt{3}} (n+\frac{1}{2})^2} \sum_m e^{-\frac{\pi t}{\sqrt{3}} (m+\frac{1}{2})^2} \right\}$$
$$= t \left\{ \vartheta_3(0, it\sqrt{3})\vartheta_3(0, \tfrac{it}{\sqrt{3}}) + \vartheta_2(0, it\sqrt{3})\vartheta_2(0, \tfrac{it}{\sqrt{3}}) \right\}.$$

Therefore,

$$\varphi_{31}(\mathcal{E}) = 2\frac{1}{\sqrt{\sqrt{3}}} \frac{1}{\sqrt{1/\sqrt{3}}} = 2.$$

These give all the topological numbers in Theorem 1.1 which we can summarize in terms of the Connes–Chern ρ-topological invariant for the field $\mathcal{E}(t)$ as

$$T_6(\mathcal{E}(t)) = (t; 3\omega - 1, \omega, 3\omega, \tfrac{1}{2}, 2).$$

The proof of Theorem 1.2 is now complete.

We end this section with computation of the topological invariants of the matrix projection of Theorem 1.2.

Topological invariants of the matrix projection e of Theorem 1.2. To compute the topological invariants of the matrix projection e, we use the values in Theorem 1.1, its covariant form (1.6), together with the following relations between the κ-traces ψ_k^θ of A_θ and the κ_0-traces $\psi_k^{\theta_q}$ of A_{θ_q} under the condition that p is even (which we assumed in Eq. (3.2))

$$\psi_0^\theta \zeta = \psi_0^{\theta_q} + \delta_3^q \psi_1^{\theta_q} + \delta_3^q \psi_2^{\theta_q} \tag{5.5}$$
$$\psi_1^\theta \zeta = e(-\tfrac{p}{3})\delta_3^{q-1} \psi_1^{\theta_q} + e(\tfrac{p}{3})\delta_3^{q-2} \psi_2^{\theta_q} \tag{5.6}$$
$$\psi_2^\theta \zeta = e(\tfrac{p}{3})\delta_3^{q-2} \psi_1^{\theta_q} + e(-\tfrac{p}{3})\delta_3^{q-1} \psi_2^{\theta_q} \tag{5.7}$$

where $\zeta = \zeta_{q,\theta}$ was defined by (3.7). (These are quickly verified by evaluating both sides on the basis elements $U_{\theta_q}^m V_{\theta_q}^n$.) One therefore obtains the κ-topological invariants of e to be

$$\psi_0^\theta(e) = (1 + 2\delta_3^q)\omega, \qquad \psi_1^\theta(e) = \psi_2^\theta(e) = [e(-\tfrac{p}{3})\delta_3^{q-1} + e(\tfrac{p}{3})\delta_3^{q-2}]\omega. \tag{5.8}$$

To obtains its ρ-topological invariants, we similarly check (based on the assumption that p is even, as θ was taken in the class $\mathbb{G}$)

$$\varphi_1^\theta \zeta = \varphi_1^{\theta_q}, \qquad \varphi_{30}^\theta \zeta = \varphi_{30}^{\theta_q}, \qquad \varphi_{31}^\theta \zeta = \varphi_{31}^{\theta_q}$$
$$\varphi_{20}^\theta \zeta = (1 - \delta_3^q)\varphi_{20}^{\theta_q} + \delta_3^q \varphi_{21}^{\theta_q}, \qquad \varphi_{21}^\theta \zeta = (1 - e(-\tfrac{pq}{3}))\varphi_{20}^{\theta_q} + e(-\tfrac{pq}{3})\varphi_{21}^{\theta_q}.$$

These give us the values

$$\varphi_1^\theta(e) = 3\omega - 1, \qquad \varphi_{20}^\theta(e) = (1 + 2\delta_3^q)\omega, \qquad \varphi_{21}^\theta(e) = (1 + 2e(-\tfrac{pq}{3}))\omega,$$
$$\varphi_{30}^\theta(e) = \frac{1}{2}, \qquad \varphi_{31}^\theta(e) = 2.$$

(We note that the values for φ_{20} and φ_{21} here are consistent with Eq. (1.12).)

Acknowledgements

The author thanks the referee for helpful comments and questions. This research work has been partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] O. Bratteli, G.A. Elliott, D.E. Evans, A. Kishimoto, Non-commutative spheres II: rational rotation, J. Oper. Theory 27 (1992) 53–85.
[2] O. Bratteli, A. Kishimoto, Non-commutative spheres III. Irrational rotations, Commun. Math. Phys. 147 (1992) 605–624.
[3] J. Buck, S. Walters, Connes–Chern characters of hexic and cubic modules, J. Oper. Theory 57 (2007) 35–65.
[4] J. Buck, S. Walters, Non-commutative spheres associated with the hexic transform and their K-theory, J. Oper. Theory 58 (2) (2007) 101–122.
[5] A. Connes, Noncommutative Geometry, Academic Press, 1994.
[6] A. Connes, M. Douglas, A. Schwarz, Noncommutative geometry and matrix theory: compactification on tori, J. High Energy Phys. 2 (1998) 003, arXiv:hep-th/9711162.
[7] S. Echterhoff, W. Lück, N.C. Phillips, S. Walters, The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$, J. Reine Angew. Math. (Crelle's Journal) 639 (2010) 173–221.
[8] G.A. Elliott, Gaps in the spectrum of an almost periodic Schrödinger operator, C. R. Math. Rep. Acad. Sci. Canada 4 (1982) 255–259.
[9] C. Farsi, N. Watling, Quartic algebras, Can. J. Math. 44 (6) (1992) 1167–1191.
[10] C. Farsi, N. Watling, Cubic algebras, J. Oper. Theory 30 (1993) 243–266.
[11] C. Farsi, N. Watling, Elliptic algebras, J. Funct. Anal. 118 (1993) 1–21.
[12] A. Konechny, A. Schwarz, Introduction to m(atrix) theory and noncommutative geometry, Phys. Rep. 360 (2002) 353–465, arXiv:hep-th/0012145.
[13] A. Konechny, A. Schwarz, Moduli spaces of maximally supersymmetric solutions on noncommutative tori and noncommutative orbifolds, J. High Energy Phys. 9 (5) (2000), arXiv:hep-th/0005174.
[14] A. Konechny, A. Schwarz, Compactification of m(atrix) theory on noncommutative toroidal orbifolds, Nucl. Phys. B 591 (2000) 667–684, arXiv:hep-th/9912185.
[15] A. Polishchuk, Holomorphic bundles on 2-dimensional noncommutative toric orbifolds, in: C. Consani, M. Marcolli (Eds.), Noncommutative Geometry and Number Theory, Vieweg Publ., 2006, pp. 341–359.
[16] M. Rieffel, Projective modules over higher-dimensional non-commutative tori, Can. J. Math. 40 (2) (1988) 257–338.
[17] N. Seiberg, E. Witten, String theory and noncommutative geometry, J. High Energy Phys. 09 (1999) 032, 100 pp., arXiv:hep-th/9908142.
[18] S. Walters, Projective modules over the non-commutative sphere, J. Lond. Math. Soc. 51 (2) (1995) 589–602.
[19] S.G. Walters, Inductive limit automorphisms of the irrational rotation algebra, Commun. Math. Phys. 171 (1995) 365–381.
[20] S. Walters, Chern characters of Fourier modules, Can. J. Math. 52 (3) (2000) 633–672.
[21] S. Walters, K-theory of non-commutative spheres arising from the Fourier automorphism, Can. J. Math. 53 (3) (2001) 631–672.
[22] S. Walters, On Fourier orthogonal projections in the rotation algebra, J. Lond. Math. Soc. (2) 68 (1) (2003) 193–205, arXiv:math.OA/0012053.
[23] S. Walters, Periodic integral transforms and C^*-algebras, C. R. Math. Rep. Acad. Sci. Canada 26 (2) (2004) 55–61, arXiv:math.OA/0401340.
[24] S. Walters, The AF structure of non-commutative toroidal $\mathbb{Z}/4\mathbb{Z}$ orbifolds, J. Reine Angew. Math. (Crelle's Journal) 568 (2004) 139–196, arXiv:math.OA/0207239.
[25] S. Walters, On the inductive limit structure of order four automorphisms of the irrational rotation algebra, Int. J. Math. 17 (1) (2006) 1–11.
[26] S. Walters, Decomposable projections related to the Fourier and flip automorphisms, Math. Scand. 107 (2010) 174–197.
[27] S. Walters, Cubic and Hexic integral transforms for locally compact Abelian groups, C. R. Math. Rep. Acad. Sci. Canada (2014), 9 pages, in press.
[28] E. Witten, D-branes and K-theory, J. High Energy Phys. 9812 (1998) 019, arXiv:hep-th/9810188.
[29] E.T. Whittaker, G.N. Watson, A Course in Modern Analysis, 4th edition, Cambridge Univ. Press, 1950.